Mean Field Theories and Dual Variation

Takashi Suzuki

Atlantis Studies in Mathematics
for Engineering and Science
Editor: C.K. Chui

ATLANTIS STUDIES IN MATHEMATICS FOR ENGINEERING AND SCIENCE

VOLUME 2

SERIES EDITOR: **C.K. CHUI**

Atlantis Studies in Mathematics for Engineering and Science

Series Editor:

C. K. Chui, Stanford University, USA

(ISSN: 1875-7642)

Aims and scope of the series

The series 'Atlantis Studies in Mathematics for Engineering and Science (AMES) publishes high quality monographs in applied mathematics, computational mathematics, and statistics that have the potential to make a significant impact on the advancement of engineering and science on the one hand, and economics and commerce on the other. We welcome submission of book proposals and manuscripts from mathematical scientists worldwide who share our vision of mathematics as the engine of progress in the disciplines mentioned above.

All books in this series are co-published with World Scientific.

For more information on this series and our other book series, please visit our website at:

www.atlantis-press.com/publications/books

ATLANTIS PRESS

AMSTERDAM – PARIS

World Scientific

Mean Field Theories and Dual Variation

Takashi SUZUKI

Division of Mathematical Science
Department of Systems Innovation
Graduate School of Engineering Science
Osaka University
Osaka
Japan

ATLANTIS PRESS

AMSTERDAM – PARIS

World Scientific

Atlantis Press

29, avenue Laumière
75019 Paris, France

For information on all Atlantis Press publications, visit our website at: *www.atlantis-press.com*

ISBN: 978-90-78677-14-7 e-ISBN: 978-94-91216-22-0
ISSN: 1875-7642

Preface

In general, systems of nonlinear partial differential equations are formulated and studied individually according to sepcific physical or biological models of interest. There are, however, certain classes of Lagrangian systems for which the asymptotic behavior of the non-stationary solutions, including the blowup mechanism, is controlled by their total sets of stationary solutions. In this book, we will show that these equations have their origin in mean field theory and that the associated elliptic problems are provided with mass and energy quantizations because of their scaling properties.

Mean field approximation has been adopted to describe macroscopic phenomena from microscopic overviews, with some success, though still in progress, in many areas of science, such as the study of turbulence, gauge field, plasma physics, self-interacting fluids, kinetic theory, distribution function method in quantum chemistry, tumor growth modelling, phenomenology of critical phenomena. In this last scientific area, phase transition, phase separation, and shape memory alloys are included.

However, in spite of such a wide range of scientific areas that are concerned with the mean field theory, a unified study of its mathematical structure has not been discusssed explicitly in the open literature. The benefit of this point of view on nonlinear problems should have siginificant impact on future research, as will be seen from the underlying features of self-assembly or bottom-up self-organization which is to be illustrated in a unified way. The aim of this book is to formulate the variational and hierarchical aspects of the equations that arise in the mean field theory from macroscopic profiles to microscopic principles, from dynamics to equilibrium, and from biological models to models that arise from chemistry and physics.

One of the key concepts to be used repeatedly in our discussion in this book is the notion of duality, which has been the origin of the extension of functions to distributions[273] that provides a useful tool for the formulation of *weak solutions* in the theory of partial differential equations. In addition, Riesz' representation theorem[100] provides the formulation of duals for such important function spaces as $C(K)' \cong \mathscr{M}(K)$ and $L^1(\Omega)' \cong L^\infty(\Omega)$, where K and Ω are compact and open sets in $\mathbf{R}^n$, respectively. Thus if a family of continuous approximate solutions is not compact but is provided with *a priori* estimates, say, in the L^∞ or L^1 norm, then the weak solution is realized as an L^∞ function or a measure, respectively, whereby several fundamental profiles caused by the nonlinearity can be observed (e.g., interface, shock, localization, concentration, condensate, blowup, self-similar evolution, travelling and spiral waves, ...). Another concept of duality, namely, that of Hardy-BMO[288] is reflexive. It is represented by $L\log L(\Omega)' \cong Exp(\Omega)$ and $Exp(\Omega)' \cong L\log L(\Omega)$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain. The Hardy-BMO duality is associated with the mean field hierarchy through the *entropy* functional and the *Gibbs measure*. Thus this duality notion can be applied to the study of phenomena between particles and fields in the microscopic level as governed by the Boltzmann principle and the zeta-function, respectively. Yet another duality concept that will be used in this book is operator theoretical. For example, for some given rectangular matrix A a column vector b, the duality of the linear model $Ax = b$ is the transposed operation that arises in the context of the Fredholm formulation concerning the existence and uniqueness of the solution. On the other hand, the linear inequality $Ax \geq b$ in convex analysis has an analogous duality formulation. For example, this formulation results from integrating a linear functional equation where the *Legendre transformation* takes on the role of the transposed operator.[30] This involutive operation is often referred to as the Fenchel-Moreau duality, which results in the more complicated dualities of Kuhn-Tucker and Toland. Here, the *Toland* duality is concerned with the functional represented by the difference of two convex functionals, such as the Helmholtz free energy. The field functional then is obtained in its dual form, and these two principal functionals are regarded as unfoldings of the *Lagrangian*.

This book consists of two main chapters. The first chapter is devoted to the study of variational structures, where the quantized blowup mechanism observed in the chemotaxis system[304] is first described. Within this section, the free energy transmission, which results in the formation of self-assembly, is emphasized which means that the source of the emergence is the wedge of blowup envelope, while entropy and mass are exchanged to create a clean self with the quantized mass. The leading principle derived from this study can be summarized as follows: dual variation sealed in the mean field hierarchy and scaling that reveals this hierarchy. In fact, the phenomenon of quantization is sealed in the total set of stationary solutions, referred to as the nonlinear spectral mechanics. The structure of dual variation controls, not only the local dynamics around the stationary state, but also the formation of self-assembly realized as a global dynamics including the blowup of the solution. This duality is associated with the convexity of the functional; and thus, our description involves convex analysis as mentioned above. We will examine the phenomena and logistics of both variational and dynamical structures of several problems including phase transition, phase separation, hysteresis in shape memory alloys which review the theory of non-equilibrium thermodynamics.

Chapter 2 is devoted to the discussion of the method of scaling for revealing the mathematical principles hidden in the mean field hierarchy subject to the second law of thermodynamics. More precisely, we will describe the quantized blowup mechanism observed in self-gravitating fluid, turbulence, and self-dual gauge field. In particular, blowup anlaysis, one of the most important applications of the method of scaling, is used to clarify this mechanism. This analysis consists of four ingredients, namely, scaling invariance of the problem, classification of the solution of the limit problem defined on the whole space and time, control of the rescaled solution at infinity, and hierarchical arguments. Meanwhile we will also describe the method of statistical mechanics for the study of macro-scopic phenomena caused by the micro-scopic principles. In particular, we will re-formulate the simplified system of chemotaxis as a fundamental equation of the material transport, the Smoluchowski-Poisson equation.

Unfortunately, a complete overview of the topics mentioned above is beyond the scope of this book, and the list of references is far from being sufficient. Furthermore, several important related topics are missing, among which are the complex Ginzburg-Landau free energy associated with super-conductivity in low temperature,[22,227] Arnold's variational principle for describinig the MHD equilibrium,[11] the density function theory of quantum chemistry,[43] kinetic equations in fluid dynamics,[29] and the duality between Aleksandrov's problem in integral geometry and the optimal mass transport theory.[238] There are also several stationary problems in engineering associated with duality[113] which may be extended to the non-stationary problems of their own.

We take this opportunity to clarify several terminologies used in this book that are the concern of thermal phenomena associated with *dissipation*. First, we follow the classical concepts of equilibrium thermodynamics and classify thermal systems into isolated, closed, and open systems. This classification is also associated with the theory of equilibrium statistical mechanics. More precisely, these systems are provided with the microscopic structures of micro-canonical, canonical, and grand-canonical ensembles, respectively. A closed system here indicates the lack of transport of materials between the outer systems, whereby the transport of heat and that of the energy are permitted. In the open system, on the other hand, the transport of material between the outer systems is also permitted. Next, in view of the theory of non-equilibrium thermodynamics, we add one more aspect to the openness, that is the dissipation of entropy to the outer system.[224] Under this agreement, closedness is re-formulated as a system provided with decreasing total free energy or that provided with increasing total entropy. Thus we have two kinds of closedness: physical closedness in terms of kinetic and/or material, and thermodynamical closedness described by two typical models discussed in details in this book. Free energy *transmission* occurs in the latter case and could be the origin of "self-assembly".

In this connection, we note that the *dissipative system* is defined by the presence of an attractor in the theory of dynamical systems. Gradient system with compact semi-orbits is a typical dissipative system in this case.[128] We believe, however, that this definition of dissipative system does not describe what were observed by.[224] More precisely, decrease of the free energy or increase of the entropy means thermodynamical *closedness*. It is formulated by the gradient system in the same two models as mentioned above, and, therefore, is never provided with the dissipation in the sense of non-equilibrium thermodynamics[224] in spite of the fact that these two models are typical dissipative systems in the theory of dynamical systems. To avoid this confusion, the above mentioned terminology of dissipative system in the theory of dynamical systems[128] is *not* used in this book. Actually, a recent paradigm asserts two aspects of self-organization, *far-from-equilibrium* (top-down self-organization) and *self-assembly* (bottom -up self-organization), emphasizing the role of their *hierarchical developments*, with the Hopf bifurcation casting the threshold.[345] Dissipation occurs in a state far-from-equilibrium, which results in a spiral wave, a travelling wave, a periodic structure, a sef-similar development, and so forth; while self-assembly, we believe, is formed near-from-equilibrium of thermodynamically closed system induced by the stationary states, developing a condensate, a collapse, a blowup, a spike, and so forth. This book is concerned with the formation of self-assembly. Its profile is provided with the ”triple seal,” of which basic concepts have already been described. First, several features of self-assembly that is one aspect of the self-organization are sealed in thermodynamically closed systems. Secondly, the dynamics of the closed system is sealed in the total set of stationary states. Finally, the stationary states themselves are sealed in the (skew-) Lagrangian, provided with the structure of dual variation. These features of the mean field hierarchy are certainly revealed by the method of scaling derived from the microscopic principle. In summary, formation of self-assembly arises in thermodynamically closed systems, and hence this process is subject to the calculus of variation. The closed system follows the microscopic principle. This principle controls the mean field hierarchy totally, and, therefore, fundamental profiles of the macroscopic mean field equation are revealed by the method of scaling. We would like to thank Professor Tomohiko Yamaguchi for several stimulative discussions on non-equilibrium thermodynamics. Thanks are also due to Professors Biotr Biler, Fumio Kikuchi, Futoshi Takahashi, and Gershon Wolansky for careful reading over the primary version of the manuscript.

Takashi Suzuki
Osaka, Japan
June, 30, 2008

Contents

Chapter 1

Duality - Sealed Variation

Self-organization is a phenomenon widely observed in physics, chemistry, and biology. Usually, it is mentioned in the context of far-from-equilibrium of dissipative open systems describing the discharge of entorpy, but the other aspect, formation of self-assembly, is sealed in the total set of stationary solutions of thermodynamically closed systems. This stationary problem is contained in a hierarchy of the mean field of many self-interacting particles, whereby the transmission of free energy is a leading factor. We can observe a unified structure of calculus of variation in these mathematical models, and obtain a guideline of the study from it, especially the total structure of stationary solutions and its roles in dynamics.
Equations describing self-assembly have been classified into models (A), (B), and (C). Although model (C) equations indicate models other than those of model (A) or (B) equations, see,[96] some of them are fomulated as the (skew-) gradient system with semi-duality. In this chapter, first, we take a simplified system of chemotaxis to confirm the above story, and then develop the abstract theory of Toland duality. Next, we examine the closed structures of model (A) and model (B) equations, and, finally, confirm the profiles of dual variation sealed in several model (C) equations.

1.1. Chemotaxis

Chemotaxis is a feature that some living things are attracted by special chemical source, which results in, for example, the formation of spores in the case of cellular slime molds. In more details, under sufficient food and water, the spores of cellular slime molds break and small amoebas are born. First, the dissipative structure makes them quite active, but they are eventually combined with spores again after the first twenty-four hours. At this occasion, the chemotactic feature to the chemical substances secreted takes a role. This process is made up of a mathematical model, and then formation of the delta-function singularity is conjectured. The effect of chemotaxis is formulated also in the material transport theory subject to the total mass conservation and the decrease of the free energy. The quantized blowup mechanism to such a system is actually proven in this context, using the weak formulation made by the duality and the scaling invariance of the problem. This section illstrates the story of the proof of such quantized blowup mechanics, its backgrounds, and related topics such as the tumor growth model and the geometric flow.

T. Suzuki, *Mean Field Theories and Dual Variation*, Atlantis Studies in Mathematics for Engineering and Science 2, DOI 10.1007/978-94-91216-22-0_1, © 2008 Atlantis Press/World Scientific

1.1.1. Thermal Transport

If $\Omega \subset \mathbf{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary $\partial\Omega$ and $v = v(x)$ is a C^1 vector field defined on $\overline{\Omega}$, then the divergence formula of Gauss holds as

$$\int_\Omega \nabla \cdot v dx = \int_{\partial\Omega} \nu \cdot v dS, \tag{1.1}$$

where $\cdot$ is the inner-product in $\mathbf{R}^n$,

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

the gradient operator,

$$\nu = \begin{pmatrix} \nu^1 \\ \cdots \\ \nu^n \end{pmatrix}$$

the outer normal vector, and dS the area (or line) element.
Thus

$$\nabla \cdot v = \frac{\partial v^1}{\partial x_1} + \ldots + \frac{\partial v^n}{\partial x_n}$$

is the divergence of the vector field v, indicated by

$$v = \begin{pmatrix} v^1 \\ \cdots \\ v^n \end{pmatrix}.$$

Furthermore,

$$\nu = \begin{pmatrix} x_2'(s) \\ -x_1'(s) \end{pmatrix}$$
$$dS = ds$$

if $n = 2$ and $\partial\Omega$ is parametrized as $x = (x_1(s), x_2(s))$ by the arc-length parameter s:

$$\sqrt{x_1'(s)^2 + x_2'(s)^2} = 1$$

and

$$\nu = \frac{x_u \times x_v}{|x_u \times x_v|}$$
$$dS = |x_u \times x_v| dudv = \sqrt{EG - F^2} dudv$$

if $n = 3$ and $\partial\Omega$ is parametrized as $x = x(u,v) \in \mathbf{R}^3$ by $(u,v) \in \mathbf{R}^2$, where

$$x_u = \frac{\partial x}{\partial u}, \qquad x_v = \frac{\partial x}{\partial v}$$
$$E = |x_u|^2, \quad F = x_u \cdot x_v, \quad G = |x_v|^2$$

and $\times$ denotes the outer product in $\mathbf{R}^3$.
From the classical theory of ordinary differential equations, we can define a local flow $\{T_t\}$ by $x(t) = T_t x_0$, where $x = x(t)$ is a unique solution to

$$\frac{dx}{dt} = v(x), \qquad x|_{t=0} = x_0 \in \mathbf{R}^n. \tag{1.2}$$

Then, if $\omega \subset \mathbf{R}^n$ is a bounded domain, the set

$$T_t(\omega) = \{T_t x \mid x \in \omega\}$$

indicates the region occupied by the particles at $t = t$ that have moved from those in ω at $t = 0$, subject to the velocity field v. Its volume (or area) is equal to

$$|T_t(\omega)| = \int_{T_t(\omega)} d\xi = \int_\omega |J_t(x)|\, dx$$

using the transformation of variables $\xi = T_t x$, where

$$J_t(x) = \det\left(\frac{\partial \xi_i}{\partial x_j}\right)_{1\le i,j\le n}$$

denotes the Jacobian.
We can derive, see,[282]

$$\left.\frac{\partial \xi_i}{\partial x_j}\right|_{t=0} = \delta_{ij}, \qquad \left.\frac{d}{dt}\frac{\partial \xi_i}{\partial x_j}\right|_{t=0} = \frac{\partial v^i}{\partial x_j}$$

or, equivalently,

$$\frac{\partial \xi_i}{\partial x_j} = \delta_{ij} + t\frac{\partial v^i}{\partial x_j} + o(t)$$

from (1.2), and, therefore,

$$J_t(x) = 1 + t\left(\frac{\partial v^1}{\partial x_1} + \ldots + \frac{\partial v^n}{\partial x_n}\right) + o(t).$$

This relation implies

$$|T_t(\omega)| = |\omega| + t\int_\omega \nabla\cdot v dx + o(t),$$

or, equivalently,

$$\left.\frac{d}{dt}|T_t(\omega)|\right|_{t=0} = \int_\omega \nabla\cdot v dx.$$

Thus the left-hand side of (1.1) indicates the total amount of the particles flowing into Ω per unit time from inside source, which is equal to the total integral of the normal component of the velocity v on the boundary indicated by the right-hand side.
For the proof of (1.1), we approximate Ω by Ω_h using small rectangles, for example, and reduce it to

$$\int_{\Omega_h} \nabla\cdot v dx = \int_{\partial\Omega_h} \nu\cdot v dS. \tag{1.3}$$

To prove (1.3), next, we take the rectangles $\{\Omega_h^i\}_i$ composing Ω, and show

$$\int_{\Omega_h^i} \nabla \cdot v dx = \int_{\partial\Omega_h^i} \nu \cdot v dS \tag{1.4}$$

for each i. Then, $\sum_i$ of the left-hand side is equal to that of (1.3). The same is true for the right-hand side because the inner boundary integrals cancel. Thus (1.4) implies (1.3). Equality (1.4), on the other hand, is proven by changing the triple (double) integral of the left-hand side to the successive integral, and then using the fundamental theorem of differentiation and integration.
Divergence formula of Gauss (1.1) is equivalent to

$$\int_\Omega \frac{\partial f}{\partial x_i} dx = \int_{\partial\Omega} \nu_i f dS, \qquad 1 \leq i \leq n,$$

where $f = f(x)$ is a C^1 scalar field. This formula implies the integration by parts,

$$\int_\Omega w \frac{\partial g}{\partial x_i} dx = \int_{\partial\Omega} w \nu_i g dS - \int_\Omega \frac{\partial w}{\partial x_i} g dx, \qquad 1 \leq i \leq n, \tag{1.5}$$

where $w = w(x)$ and $g = g(x)$ are C^1 scalar fields.
Direction derivative of f toward ν, on the other hand, is defined by

$$\frac{\partial f}{\partial \nu} = \frac{d}{ds} f(\cdot + s\nu)\Big|_{s=0}.$$

Using

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix},$$

then we obtain

$$\frac{\partial f}{\partial \nu} = \nu \cdot \nabla f$$

by

$$\begin{aligned} \frac{\partial f}{\partial \nu}(x) &= \frac{d}{ds} f(x_1 + s\nu^1, \ldots, x_n + s\nu^n)\Big|_{s=0} \\ &= \frac{\partial f}{\partial x_1}(x)\nu^1 + \ldots \frac{\partial f}{\partial x_n}(x)\nu^n \end{aligned}$$

which means that

$$f(x + s\nu) = f(x) + s\nu \cdot \nabla f(x) + o(s)$$

as $s \to 0$ and, therefore, $\nabla f(x)$ has the direction where $f(x)$ obtains the infinitesimally maximum increase, and the length of this vector is equal to the degree of the variation of $f(x)$ in this direction.[282]

Applying (1.1) for $v = \nabla f$, we obtain

$$\int_\Omega \Delta f dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} dS,$$

using Laplacian,

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

provided that f is a C^2 scalar field. Similarly, putting $w = \frac{\partial f}{\partial x_i}$ and taking $\sum_i$ in (1.5), we obtain

$$\int_\Omega \nabla g \cdot \nabla f dx = \int_{\partial\Omega} g \frac{\partial f}{\partial \nu} dS - \int_\Omega g \Delta f dx$$

if g and f are C^1 and C^2 scalar fields, respectively. This equality implies Green's formula

$$\int_\Omega (g\Delta f - f\Delta g) dx = \int_{\partial\Omega} \left(g\frac{\partial f}{\partial \nu} - f\frac{\partial g}{\partial \nu} \right) dS$$

valid to C^2 scalar fields indicated by $f = f(x)$ and $g = g(x)$.

Heat Equation

If Ω is occupied with a heat conductor and ω is a sub-domain with C^1 boundary $\partial\omega$, then

$$\int_\omega c\rho\theta_t dx$$

denotes the infinitesimal change of the heat energy inside ω, where c, ρ, and θ denote the specific heat, the density, and the temperature, respectively. Then, from Newton-Fourier-Fick's law, the heat energy lost per unit time is proportional to the temperature gradient, and thus we obtain

$$\int_\omega c\rho\theta_t dx = \int_{\partial\omega} \kappa \frac{\partial \theta}{\partial \nu} dS \tag{1.6}$$

with the conductivity κ.
The right-hand side of (1.6) is equal to

$$\int_{\partial\omega} \nu \cdot (\kappa \nabla\theta) dS = \int_\omega \nabla \cdot (\kappa\nabla\theta) dx$$

by (1.1), and hence it holds that

$$c\rho\theta_t = \nabla \cdot (\kappa\nabla\theta) \tag{1.7}$$

called the heat equation because ω is arbitrary.

1.1.2. Collapse Formation

Simplified system of chemotaxis[157,209] has several backgrounds in statistical mechanics, thermodynamics, and biology.[304] For this system, we have a remarkable profile of the solution, called the quantized blowup mechanism.[304,305]
In the context of biology, it is associated with the chemotactic feature of cellular slime molds,[169] typically formulated by

$$\begin{aligned} & u_t = \nabla \cdot (\nabla u - u\nabla v) \\ & -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\ & \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\ & \int_\Omega v = 0, \end{aligned} \tag{1.8}$$

where $\Omega \subset \mathbf{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outer unit normal vector.
In this case, $u = u(x,t)$ stands for the density of cellular slime molds, and the first equation of (1.8) is written as

$$u_t = -\nabla \cdot j$$

with

$$j = -\nabla u + u\nabla v \tag{1.9}$$

which means that

$$\frac{d}{dt}\int_\omega u = -\int_{\partial\omega} \nu \cdot j$$

for any sub-domain ω with C^1 boundary $\partial\omega$ by (1.1). Thus, the vector field j stands for the flux of u, and the first equation of (1.8) describes mass conservation. The null flux condition,

$$\nu \cdot j = 0,$$

therefore, is imposed on the boundary in (1.8), which guarantees the total mass conservation,

$$\|u(\cdot,t)\|_1 = \|u_0\|_1 \tag{1.10}$$

for $u_0 = u(\cdot,0) \geq 0$ because $u = u(x,t) \geq 0$ follows from the maximum principle.[259] Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm:

$$\|f\|_p = \begin{cases} \{\int_\Omega |f|^p\}^{1/p}, & 1 \leq p < \infty \\ ess.\sup_{x\in\Omega} |f(x)|, & p = \infty. \end{cases}$$

The component $v = v(x,t)$ denotes the concentration of a chemical substance secreted by the slime molds. Its production law is described by the second equation which constitutes

the Poisson equation. We shall examine its details in the next paragraph. Roughly, since ∇v stands for the gradient created by the particle density u, the second equation indicates that u acts as a source of v, namely, the self-attractive gradient field ∇v is created by u. The form (1.9) of j, on the other hand, indicates that v acts as a carrier of u, formulated by the phenomenological relation, see §1.4.4. Thus, the term $u\nabla v$ describes the chemotactic feature of the cellular slime molds, and u is carried proportionally to the gradient ∇v of v. This chemotaxis term, furthermore, competes the diffusion $-\nabla u$ of u for u to vary.
System (1.8) is *well-posed* locally in time.[24,282,304,344] If the initial value $u_0 \geq 0$ of u is sufficiently regular, then there is a unique (classical) non-negative solution locally in time, and, therefore, if $T_{\max}$ denotes the supremum of its existence time, then we obtain $T_{\max} > 0$. In case $T = T_{\max} < +\infty$, this T is called the *blowup time*, and actually, there arises

$$\lim_{t\uparrow T} \|u(\cdot,t)\|_\infty = +\infty.$$

For this system (1.8), we have the following theorems.[279,304]

Theorem 1.1.
If $n = 2$ and $T = T_{\max} < +\infty$, then it holds that

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathscr{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{1.11}$$

as $t \uparrow T$, where

$$\mathscr{S} = \{x_0 \in \overline{\Omega} \mid \text{there exists } (x_k,t_k) \to (x_0,T) \\ \text{such that } u(x_k,t_k) \to +\infty\} \tag{1.12}$$

denotes the blouwp set of u and

$$0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathscr{S}).$$

Theorem 1.2. *We have*

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases} \tag{1.13}$$

and hence

$$2\sharp(\mathscr{S}\cap\Omega) + \sharp(\mathscr{S}\cap\partial\Omega) \leq \|u_0\|_1/(4\pi) \tag{1.14}$$

by (1.10).

The convergence (1.11) is $*$-weakly in the set of measures on $\overline{\Omega}$, denoted by $\mathscr{M}(\overline{\Omega})$, which means

$$\lim_{t\uparrow T} \int_\Omega u(\cdot,t)\varphi = \sum_{x_0\in\mathscr{S}} m(x_0)\varphi(x_0) + \int_\Omega f\varphi$$

for any $\varphi \in C(\overline{\Omega})$. This phenomenon of the appearence of the delta-function singularity in $u(x,t)dx$ is called the formation of *collapse*, and the equality $m(x_0) = m_*(x_0)$ is referred to as the *mass quantization*.

In 1970, Keller-Segel[169] proposed a system of parabolic equations with a more complicated chemotaxis term competing to the diffusion. System (1.8) is a simplified form introduced in 1992 by Jäger-Luckhaus.[157] In 1973, Nanjundiah[217] conjectured the formation of collapse, and Childress-Percus[61] refined this conjecture in 1981 by heuristic arguments. More precisely, formation of collapse is supposed to hold only in the case of $n = 2$, and 8π is expected to be the threshold of $\lambda = \|u_0\|_1$ for the existence of the solution global in time. The latter means that $\lambda < 8\pi$ will imply $T_{\max} = +\infty$, while any $\lambda > 8\pi$ admits $u_0 \geq 0$ such that $\|u_0\|_1 = \lambda$ and $T_{\max} < +\infty$.

In 1995, Nagai[209] showed that this 8π threshold conjecture is affirmative for radially symmetric solutions, but later, 4π is proven to be the real threshold value of non-radially symmetric case.[24,108,210,211,278] The above mentioned Theorems 1.1 and 1.2 are regarded as a localization of this threshold combined with the formation of delta-fucntion sigularity. Concerning the other space dimensions, on the other hand, it is proven that there is no blowup for $n = 1$ and no L^1 threshold blowup for $n = 3$, see.[140,141,209,275] In the case of $n = 2$, however, the formation of collapse is valid even to perturbed systems.[176]

1.1.3. Mean Field Hierarchy

System (1.8) arises also in physics. In this context, we call it the *Smoluchowski - Poisson equation*, whereby $u = u(x,t)$ indicates the distribution function of self-gravitating particles, and $v = v(x,t)$ is the gravitational field created by them. In more details, the second equation of (1.8) is equivalent to

$$v(x,t) = \int_\Omega G(x,x')u(x',t)dx',$$

where $G = G(x,x')$ denotes the Green's function associated with the elliptic boundary value problem called the Poisson equation

$$\begin{aligned} &-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \\ &\int_\Omega v = 0, \end{aligned} \tag{1.15}$$

and that this Green's function behaves like the gravitational potential,

$$G(x,x') \approx \Gamma(x-x')$$

for

$$\Gamma(x) = \begin{cases} \frac{1}{4\pi} \cdot \frac{1}{|x|}, & n = 3 \\ \frac{1}{2\pi} \log \frac{1}{|x|}, & n = 2. \end{cases}$$

This equation thus describes the formation of gravitational field by the particle density.

Actually, system (1.8) is contained in a hierarchy of the mean fields of many self-gravitating particles subject to the second law of thermodynamics obtained by the fluctuation-friction approach, see §2.2.5. More precisely, it is a macroscopic description of this mean field associated with the micoscopic *Langevin equation*, and the mezoscopic *Fokker-Planck-Poisson equation*.[20,338,339] This hierarchy of mean fields is governed by the decrease of Helmholtz' free energy $\mathscr{F}$, which is defined by the inner energy minus entropy if the temperature is normalized to 1, that is,

$$\mathscr{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx' \tag{1.16}$$

because $\mu(dx,t) = u(x,t)dx$ stands for the particle density, and $G = G(x,x')$ casts the potential, see §1.4.2. Thus, the first and the second terms of (1.16) indicates (-1) times entropy and the inner (kinetic) energy, respectively. The inner force is self-attractive in this case, and actually, $-\frac{1}{2}$ factor appears in the second term because of Newton's third law of action-reaction. This potential is thus symmetric,

$$G(x',x) = G(x,x'), \tag{1.17}$$

and there are several microscopic and kinetic derivations of (1.8), see §§2.2.5 and 2.2.6. The first variation $\delta\mathscr{F}(u)$ of $\mathscr{F}(u)$ is defined by

$$\langle w, \delta\mathscr{F}(u)\rangle = \frac{d}{ds}\mathscr{F}(u+sw)\Big|_{s=0}. \tag{1.18}$$

Identifying this paring $\langle\ ,\ \rangle$ with the L^2 inner product, we obtain

$$\delta\mathscr{F}(u) = \log u - v$$

for

$$v = \int_\Omega G(\cdot,x')u(x')dx'.$$

Thus (1.8) reads;

$$\begin{aligned} &u_t = \nabla\cdot(u\nabla\delta\mathscr{F}(u)) && \text{in } \Omega\times(0,T) \\ &u\frac{\partial}{\partial\nu}\delta\mathscr{F}(u) = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{1.19}$$

This problem is a form of the *model B* equation[96,130,146,252] derived from the free energy $\mathscr{F} = \mathscr{F}(u)$, and, consequently, it follows that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u = -\int_{\partial\Omega} u\frac{\partial}{\partial\nu}\delta\mathscr{F}(u) = 0 \\ &\frac{d}{dt}\mathscr{F}(u) = -\int_\Omega u\,|\nabla\delta\mathscr{F}(u)|^2 \le 0, \end{aligned} \tag{1.20}$$

see §1.3.1. The second inequality of (1.20) means the decrease of the free energy, while the first equality of (1.20), combined with $u = u(x,t) \ge 0$, assures the total mass conservation (1.10),

$$\lambda \equiv \|u_0\|_1 = \|u(\cdot,t)\|_1, \qquad t\in[0,T_{\max}). \tag{1.21}$$

Relation (1.21) leads to the selection $n = 2$ for the formation of collapse, using the dimension analysis.[61] In more details, if u is concenrated on a region with the radius $\delta > 0$, then it is of order δ^{-n} because of this property, (1.21). Then, we replace δ^{-1} by ∇ in (1.8), and take $\delta^{1-n/2}$ and $\delta^{1+n/2}$ for v and t, respectively, which results in

$$\delta^{-(1+3n/2)}\left(\delta^0, \delta^0, \delta^{-1+n/2}\right) = 0$$
$$\delta^{-n}\left(\delta^0, \cdot, \delta^{-1+n/2}\right) = 0$$

by

$$u_t - \nabla\cdot(u\nabla v) - \Delta v = 0$$
$$u - \frac{1}{|\Omega|}\int_\Omega u + \Delta v = 0.$$

Balance of these relations, thus, implies $n = 2$. Surprisingly, this heuristic argument is the origin of a rigorous proof of the formation of collapse and with mass quantization, that is, the *scaling* argument.

Theorem 1.2 concerned with the mass quantization actually describes the "local" L^1 threshold for the post-blowup continuation. The reasons why 8π was conjectured first and why it was modified later to 4π lie in the structure of the total set of stationary states.

1.1.4. Stationary State

As we described, the above mentioned quantized blowup mechanism of non-stationary solutions is sealed in the total set of stationary solultions, which are defined by the zero free energy consumption. More precisely, from (1.20), the stationary state of (1.8) is realized by

$$\delta\mathscr{F}(u) = \text{constant}, \qquad \int_\Omega u = \lambda$$

or, equivalently,

$$\log u - v = \text{constant}, \qquad \|u\|_1 = \lambda, \tag{1.22}$$

where $\lambda = \|u_0\|_1$ and

$$v = (-\Delta_N)^{-1}(u - \overline{u}), \qquad \int_\Omega v = 0 \tag{1.23}$$

for

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u.$$

Here and henceforth, $-\Delta_N$ denotes the Laplacian $-\Delta$ provided with the Neumann boundary condition, $\frac{\partial}{\partial \nu}\cdot\Big|_{\partial\Omega} = 0$. Eliminating u in (1.22), now we obtain the relation

$$u = \frac{\lambda e^v}{\int_\Omega e^v}, \tag{1.24}$$

and, therefore, the nonlinear eigenvalue problem with non-local term,

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \qquad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

$$\int_\Omega v = 0 \tag{1.25}$$

arises from (1.23). It is worth noting that the quantized blowup mechanism at this level is already observed in the blowup family of solutions.[234,277]
More precisely, if $\{(\lambda_k, v_k)\}_{k=1}^\infty$ is a family of solutions to (1.203) for $\lambda = \lambda_k$ and $v = v_k$, satisfying $\lambda_k \to \lambda_0 \in [0,\infty)$ and $\|v_k\|_\infty \to +\infty$, then the blowup set of $\{v_k\}$, denoted by

$$\mathscr{S} = \left\{ x_0 \in \overline{\Omega} \mid \text{there exists } x_k \to x_0 \text{ such that } v_k(x_k) \to +\infty \right\},$$

is finite and furthermore, passing through a subsequence, it holds that

$$u_k(x)dx \rightharpoonup \sum_{x_0 \in \mathscr{S}} m_*(x_0)\delta_{x_0}(dx) \tag{1.26}$$

in $\mathscr{M}(\overline{\Omega})$ as $k \to \infty$, where

$$u_k = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k}}.$$

This relatioin implies $\lambda_0 \in 4\pi\mathscr{N}$, and furthermore, it holds that

$$\nabla_x \left[m_*(x_0)K(x,x_0) + \sum_{x_0' \in \mathscr{S}\setminus\{x_0\}} m_*(x_0')G(x,x_0') \right] = 0, \quad x = x_0 \tag{1.27}$$

for each $x_0 \in \mathscr{S}$, where only tangential derivative is taken in (1.27) if $x_0 \in \partial\Omega$, and

$$K(x,x') = G(x,x') + \begin{cases} \frac{1}{2\pi}\log|x-x'|, & x \in \Omega \\ \frac{1}{\pi}\log|x-x'|, & x \in \partial\Omega \end{cases} \tag{1.28}$$

stands for the regular part of the Green's function $G(x,x')$. Corresponding to (1.26), we obtain

$$v_k(x) \to v_0(x) = \sum_{x_0 \in \mathscr{S}} m_*(x_0)G(x,x_0)$$

locally uniformly in $x \in \overline{\Omega} \setminus \mathscr{S}$ and this $v_0 = v_0(x)$ is called the singular limit of $\{v_k\}$.
The above mentioned blowup mechanism emerged in the total set of stationary solutions is associated with the structures of complex analysis and differential geometry,[303] and also the scaling invariance,[181] for example,

$$-\Delta v = e^v$$

is invariant under the transformation $v^\mu(x) = v(\mu x) + 2\log\mu$ for $\mu > 0$, see §2.3.2. The singular limit of $\{v_k\}$, $\lambda_k \to \lambda_0$ arising at the first quantized value $\lambda_0 = 4\pi$ to (1.25), on the other hand, is given by $4\pi G(\cdot,x_0)$ with $x_0 \in \partial\Omega$ satisfying

$$\left. \frac{\partial}{\partial \tau_x} K(x,x_0) \right|_{x=x_0} = 0,$$

where τ is the unit tangential vector. This singular limit is "stable," attracts the non-stationary solution in infinite time, and casts the underlying driving force to the self-assembly. In other words, this singular limit is the origin of the quantized blowup mechanism of the non-stationary state, that is, formation of collapse with the quantized mass, see.[304]

1.1.5. Localization and Symmetrization

For the moment, we are concentrated on the proof of Theorems 1.1 and 1.2, whereby $n = 2$ is always assumed. First, Theorem 1.1 of the formation of collapse is proven by localizing the following criterion concerning the existence of the solution globally in time.[24,108,211]

Theorem 1.3. *If* $\lambda = \|u_0\|_1 < 4\pi$*, then it holds that* $T_{\max} = +\infty$.

Using the *dual Trudinger-Moser inequality*

$$\inf\{\mathscr{F}(u) \mid u \geq 0,\ \|u\|_1 = 4\pi\} > -\infty \tag{1.29}$$

for the proof of the above theorem is proposed by,[304] where $\mathscr{F} = \mathscr{F}(u)$ denotes the free energy defined by (1.16). In fact, from (1.29) and the first equality of (1.20) it is easy to derive

$$\sup_{t\in[0,T_{\max})} \int_\Omega u(\log u - 1)(\cdot,t) \leq C$$

with a constant $C > 0$ in the case of $\lambda < 4\pi$, which guarantees $T_{\max} = +\infty$ with

$$\sup_{t\geq 0} \|u(\cdot,t)\|_\infty < +\infty$$

by the parabolic regularity. Later in §1.2.8, we shall show the duality between (1.29) and *Chang-Yang's inequality*,[48,49]

$$\inf\{J_{4\pi}(v) \mid v \in H^1(\Omega),\ \int_\Omega v = 0\} > -\infty, \tag{1.30}$$

where

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1).$$

Here and henceforth, $H^m(\Omega) = W^{m,2}(\Omega)$ with $W^{m,p}(\Omega)$ standing for the standard Sobolev space, where $m = 0, 1, \cdots$ and $1 \leq p \leq \infty$:

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega),\ \ |\alpha| \leq m\}.$$

We shall write $W_0^{m,p}(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. Chang-Yang's inequality is relative to the *Moser-Onofri inequality*,[206,325]

$$\inf\{J_{8\pi}(v) \mid v \in H_0^1(\Omega)\} > -\infty, \tag{1.31}$$

see also §2.2.4. The total mass $\lambda = 4\pi$ in (1.30) is a half of that in (1.31) because of the concentration on the boundary of the minimizing sequence for the former. The same profile occurs to the solution to (1.8) blowing-up in finite time and this profile provides the reason why $\lambda < 4\pi$ is sharp in Theorem 1.3.

The proof of Theorem 1.1 is based on the methods of *localization* and *symmetrization*, and the details are the the following.

(1) Using nice cut-off functions, we show the formation of collapse at each *isolated* blowup point.

(2) The Gagliardo-Nirenberg inequality, see,[1] guarantees ε-regularity. In more detail, there is an absolute constant, denoted by $\varepsilon_0 > 0$, such that

$$\lim_{R\downarrow 0} \limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \Rightarrow x_0 \notin \mathscr{S}$$

which means that

$$x_0 \in \mathscr{S} \Rightarrow \limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0 \tag{1.32}$$

for any $R > 0$.

(3) If we can replace $\limsup_{t\uparrow T_{\max}}$ by $\liminf_{t\uparrow T_{\max}}$ in (1.32), then we obtain $\sharp\mathscr{S} < +\infty$ because the total mass is conserved by (1.21). In this case, any blowup point is shown to be isolated, and the formation of collapse, (1.11), is proven with $m(x_0) \geq m_*(x_0)$ using the *localized* free energy.

(4) The above replacement is justified by the weak formulation of (1.8),

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(\cdot,t)\varphi &= \int_\Omega u(\cdot,t)\Delta\varphi \\ &+\frac{1}{2}\iint_{\Omega\times\Omega} \rho_\varphi(x,x')u(x,t)u(x',t)dxdx', \end{aligned} \tag{1.33}$$

obtained by the method of symmetrization where

$$\varphi \in C^2(\overline{\Omega}), \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{1.34}$$

and

$$\begin{aligned} &\rho_\varphi(x,x') \equiv \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'} G(x,x') \\ &\in L^\infty(\Omega\times\Omega). \end{aligned}$$

This formulation is possible because of the symmetry of the Green's function, indicated by (1.17).

(5) In more detail, we obtain

$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi\right| \leq C_\varphi(\lambda+\lambda^2) \tag{1.35}$$

from (1.33), and, therefore,

$$\lim_{t\uparrow T}\int_\Omega u(\cdot,t)\varphi$$

exists for $\varphi = \varphi(x)$ in (1.34). Using this property, we can replace (1.32) by

$$x_0 \in \mathscr{S} \Rightarrow \liminf_{t \uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega \cap B(x_0,R))} \geq \varepsilon_0$$

for any $R > 0$.

Mass quantization (1.13), on the other hand, is a ”local” blowup criterion, while the global blowup criterion also follows from the weak formulation (1.33), using the second moment. A plot-type argument is given by Biler-Hilhorst-Nadzieja,[25] and we obtain the following theorem.[278,304]

Theorem 1.4. *There is an absolute constant* $\eta > 0$ *such that if we have* $x_0 \in \overline{\Omega}$ *and* $0 < R \ll 1$ *satisfying*

$$\frac{1}{R^2} \int_{\Omega \cap B(x_0,2R)} |x - x_0|^2 u_0 < \eta$$
$$\int_{\Omega \cap B(x_0,R)} u_0 > m_*(x_0),$$

then $T_{\max} < +\infty$. *More precisely, it holds that* $T_{\max} = o(R^2)$ *as* $R \downarrow 0$.

We note that the assumptions of the above theorem are extended to the case that $u_0 = u_0(x)$ is a measure, say, $u_0(x) = m(x_0)\delta_{x_0}(dx)$ with $m(x_0) > m_*(x_0)$. Based on this observation, now we infer in the following way.

(1) The argument employed in the proof of Theorem 1.4 is valid to the *weak solution*. In particular, formation of the over-quantized collapse in finite time,

$$m(x_0) > m_*(x_0)$$

in (1.11), means the *complete blowup*, that is the impossibility of the post-blowup continuation of the solution. Thus the weak (measure-valued) post-blowup continuation of the blowup solution $u = u(x,t)$ assures mass quantization,

$$m(x_0) = m_*(x_0).$$

(2) The Fokker-Planck-Poisson equation, on the other hand, admits a weak solution globally in time for appropriate initial values,[239,328] and, therefore, we use the rescaled variables to complete the proof of Theorem 1.2, regarding the hierarchy of the mean field of particles.

Although the abve described guideline is not adopted directly because of several technical difficulties, it contains key ideas for the proof of mass quantization. In the next section, we shall describe the definition of the weak solution to the rescaled case. See also §2.4.1 for the notion of complete blowup.

1.1.6. Rescaling

Fundamental concept of the method of rescaling is illustrated in §2.3.2. In the case of (1.8), we assume $T = T_{\max} < +\infty$ and $x_0 \in \mathscr{S}$ and take the standard backward self-similar variables defined by

$$y = (x - x_0)/(T-t)^{1/2}$$
$$s = -\log(T-t), \qquad t < T. \tag{1.36}$$

The formal blowup rate, on the other hand, is $(T-t)^{-1/(p-1)}$ if the nonlinearity is of degree p, and $p = 2$ in this system of chemotaxis (1.8), see §1.1.8 for details.
Putting

$$z(y,s) = (T-t)u(x,t)$$
$$w(y,s) = v(x,t), \tag{1.37}$$

we obtain

$$z_s = \nabla \cdot \left(\nabla z - z\nabla(w + |y|^2/4)\right)$$
$$0 = \Delta w + z - \frac{e^{-s}\lambda}{|\Omega|} \quad \text{in } \bigcup_{s>-\log T} e^{s/2}(\Omega - \{x_0\}) \times \{s\}$$
$$\frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \bigcup_{s>-\log T} e^{s/2}(\partial\Omega - \{x_0\}) \times \{s\}. \tag{1.38}$$

Here, we use the following ingredients for the proof of Theorem 1.2:

(1) parabolic envelope.
(2) generation of the weak solution.
(3) second moment.
(4) formation of sub-collapse.

Parabolic Envelope

Taking a nice cut-off function around $x_0 \in \mathscr{S}$ with the support radius $2R > 0$ denoted by $\varphi_{x_0,R}$, we can refine (1.35) as

$$\left| \frac{d}{dt} \int_\Omega u(\cdot,t)\varphi_{x_0,R} \right| \leq C(\lambda + \lambda^2)R^{-2}$$

with a constant $C > 0$ independent of $0 < R < 1$ which implies that

$$\left| \langle \varphi_{x_0,R}, \mu(\cdot,T) \rangle - \langle \varphi_{x_0,R}, \mu(\cdot,t) \rangle \right| \leq C(\lambda + \lambda^2)R^{-2}(T-t) \tag{1.39}$$

because $u(x,t)dx = \mu(dx,t)$ is regarded as a $*$-weakly continuous function in $\mathscr{M}(\overline{\Omega})$ on $[0,T]$, that is

$$\mu = \mu(dx,t) \in C_*([0,T], \mathscr{M}(\overline{\Omega})).$$

Since $0 < R < 1$ is arbitrary in (1.39), we can put

$$R = bR(t)$$

for given $b > 0$, provided that $0 < R(t) \equiv (T-t)^{1/2} < b^{-1}$, that is

$$\left| \left\langle \varphi_{x_0, bR(t)}, \mu(\cdot, T) \right\rangle - \left\langle \varphi_{x_0, bR(t)}, \mu(\cdot, t) \right\rangle \right| \leq Cb^{-2}$$

which implies that

$$\limsup_{t \uparrow T} \left| m(x_0) - \left\langle \varphi_{x_0, bR(t)}, \mu(\cdot, t) \right\rangle \right| \leq Cb^{-2}$$

for $x_0 \in \mathscr{S}$ by

$$\mu(dx, T) = \sum_{x_0 \in \mathscr{S}} m(x_0) \delta_{x_0}(dx) + f(x)dx,$$

and, therefore,

$$\lim_{b \uparrow +\infty} \limsup_{t \uparrow T} \left| \left\langle \varphi_{x_0, bR(t)}, \mu(\cdot, t) \right\rangle - m(x_0) \right| = 0. \tag{1.40}$$

Relation (1.40) indicates that infinitely wide parabolic region concerning the backward self-similar variables, called *parabolic envelope*, contains the whole blowup mechanism.

Generation of the Weak Solution

Given $s_k \uparrow +\infty$, we have $\{s'_k\} \subset \{s_k\}$ and $\zeta(dy, s)$ such that

$$z(y, s + s'_k)dy \rightharpoonup \zeta(dy, s) \tag{1.41}$$

in $C_*(-\infty, +\infty; \mathscr{M}(\mathbf{R}^2))$. Here, 0-extension to $z(y,s)$ is taken where it is not defined. This $\zeta = \zeta(dy, s)$ satisfies supp $\zeta(\cdot, s) \subset \overline{L}$, and is a weak solution to

$$\begin{aligned} & z_s = \nabla \cdot \left(\nabla z - z \nabla \left(w + |y|^2/4 \right) \right) \qquad \text{in } L \times (-\infty, +\infty) \\ & \left. \frac{\partial z}{\partial \nu} \right|_{\partial L} = 0 \\ & \nabla w(y, s) = \int_L \nabla \Gamma(y - y') z(y', s) dy', \end{aligned}$$

where L is $\mathbf{R}^2$ and a half space with ∂L parallel to the tangent line of $\partial\Omega$ at x_0 if $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively, and

$$\Gamma(y) = \frac{1}{2\pi} \log \frac{1}{|y|}.$$

If $x_0 \in \partial\Omega$, we take the even extension of $\zeta(dy, s)$ denoted by the same symbol without confusion. Then, all the above cases are reduced to $L = \mathbf{R}^2$,

$$\begin{aligned} & z_s = \nabla \cdot \left(\nabla z - z \nabla (w + |y|^2/4) \right) \qquad \text{in } \mathbf{R}^2 \times (-\infty, +\infty) \\ & \nabla w(y, s) = \int_{\mathbf{R}^2} \nabla \Gamma(y - y') z(y', s) dy'. \end{aligned} \tag{1.42}$$

We thus obtain a full-orbit weak solution

$$\zeta = \zeta(dy,s) \in C_*(-\infty,+\infty;\mathscr{M}(\mathbf{R}^2))$$

to (1.42). This $\zeta(dy,s)$ is a Radon measure on $\mathbf{R}^2$ satisfying

$$\begin{aligned}\zeta(\mathbf{R}^2,s) &= \begin{cases} m(x_0), & x_0 \in \Omega \\ 2m(x_0), & x_0 \in \partial\Omega \end{cases} \\ &\geq 8\pi \end{aligned} \tag{1.43}$$

for each $s \in (-\infty,+\infty)$ from the parabolic envelope.
Here, we use the one-point compactification of $\mathbf{R}^2$, denoted by $\mathbf{R}^2 \cup \{\infty\}$ which is identified with the sphere

$$S^2 = \left\{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid x_1^2+x_2^2+x_3^2 = 1\right\}$$

by the stereographic projection $y_i = x_i/(1-x_3)$ $(i = 1,2)$. The following definition of the weak solution to (1.42) is sufficient for later arguments using second moment.

(1) There is $0 \leq \nu = \nu(s) \in L^\infty_*(-\infty,+\infty;\mathscr{E}')$ satisfying

$$\nu(s)|_{\mathscr{E}} = \zeta \otimes \zeta(dydy',s) \qquad \text{a.e. } s \in \mathbf{R},$$

where

$$\begin{aligned}&\mathscr{E} = \left[\left\{\rho^0_\varphi \mid \varphi \in C^2(\mathbf{R}^2 \cup \{\infty\})\right\}\right]^{L^\infty(\mathbf{R}^2\cup\{\infty\}\times\mathbf{R}^2\cup\{\infty\})} \\ &\subset L^\infty(\mathbf{R}^2 \cup \{\infty\} \times \mathbf{R}^2 \cup \{\infty\}) \\ &\rho^0_\varphi(y,y') = \left(\nabla\varphi(y) - \nabla\varphi(y')\right) \cdot \nabla\Gamma(y-y').\end{aligned}$$

(2) For each $\varphi \in C^2(\mathbf{R}^2 \cup \{\infty\})$, the mapping $s \in \mathbf{R} \mapsto \langle\varphi,\zeta(dy,s)\rangle$ is locally absolutely continuous, and it holds that

$$\frac{d}{ds}\langle\varphi,\zeta(\cdot,s)\rangle = \langle\Delta\varphi + y\cdot\nabla\varphi/2,\zeta(\cdot,s)\rangle + \frac{1}{2}\left\langle\rho^0_\varphi,\nu(s)\right\rangle_{\mathscr{E},\mathscr{E}'},$$

a.e. $s \in \mathbf{R}$.

From (1.43) and $m(x_0) \geq m_*(x_0)$, we have only to derive

$$\zeta(\mathbf{R}^2,0) \leq 8\pi, \tag{1.44}$$

using the above mentioned properties of $\zeta(dy,s)$ to complete the proof of Theorem 1.2. For this purpose, we note that Kurokiba-Ogawa[174] used the scaling transformation and obtained the non-existence of the weighted L^2 solution globally in time for the pre-scaled system in the whole space,

$$u_t = \nabla\cdot(\nabla u - u\nabla v), \;\; -\Delta v = u \qquad \text{in } \mathbf{R}^2 \times (0,T), \tag{1.45}$$

in the case that the initial mass is greater than 8π. We shall now examine whether this argument is valid to (1.42) or not.

Second Moment - Self-Similarity

Since the (rescaled) weak solution $\zeta = \zeta(dy,s)$ is global in time, method of the second moment assures that sufficient concentration at the origin of the solution implies (1.44). In more detail, we take smooth $c = c(s)$ satisfying

$$\begin{aligned} & 0 \leq c'(s) \leq 1, \qquad s \geq 0 \\ & -1 \leq c(s) \leq 0, \qquad s \geq 0 \\ & c(s) = \begin{cases} s-1, \; 0 \leq s \leq 1/4 \\ 0, \qquad s \geq 4, \end{cases} \end{aligned} \tag{1.46}$$

and then derive

$$\begin{aligned} \frac{d}{ds}\left\langle c(|y|^2)+1, \zeta(dy,s)\right\rangle \leq C\left\langle c(|y|^2)+1, \zeta(dy,s)\right\rangle \\ +\delta \hat{m}(x_0)\left\{4-\frac{\hat{m}(x_0)}{2\pi}\right\} \qquad \text{a.e. } s \in \mathbf{R} \end{aligned} \tag{1.47}$$

with $C > 0$ and $\delta > 0$, provided that

$$\hat{m}(x_0) \equiv \zeta(\mathbf{R}^2, 0) > 8\pi.$$

Thus, there is $\delta > 0$ such that

$$\left\langle c(|y|^2)+1, \zeta(dy,0)\right\rangle < \delta \tag{1.48}$$

implies

$$\left\langle c(|y|^2)+1, \zeta(dy,s)\right\rangle < 0$$

for $s \gg 1$, a contradiction.

In,[304] it is proposed to use a self-similarity of the problem (1.42) to remove the concentration condition (1.48). This similarity arises because equation (1.42) is obtained by the following process:

(1) equation (1.8) with (potentially) self-similarity.
(2) equation (1.38) obtained by the backward self-similar transformation.
(3) weak limit as $s'_k \uparrow +\infty$ defined on the whole space-time (1.42).

To detect this property, here we take the transformation

$$\begin{aligned} & z(y,s) = e^{-s}A(y',s'), \;\; w(y,s) = B(y',s') \\ & y' = e^{-s/2}y, \;\; s' = -e^{-s}. \end{aligned} \tag{1.49}$$

Then, we obtain the same form to (1.45),

$$\begin{aligned} & A_{s'} = \nabla' \cdot \left(\nabla' A - A \nabla' B\right) \\ & \nabla' B(\cdot, s') = \int_{\mathbf{R}^2} \nabla\Gamma(\cdot - y')A(y',s')dy' \;\; \text{in } \mathbf{R}^2 \times (-\infty, 0), \end{aligned} \tag{1.50}$$

provided with the self-similarity

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \qquad v_\mu(x,t) = v(\mu x, \mu^2 t).$$

This property to (1.50) induces the self-similar transformation of (1.42):

$$
\begin{aligned}
&z(y,s) = e^{-s}A(y',s'), \;\; w(y,s) = B(y',s') \\
&z^{\mu}(y,s) = e^{-s}A_{\mu}(y',s'), \;\; w^{\mu}(y,s) = B_{\mu}(y',s') \\
&A_{\mu}(y',s') = \mu^2 A(\mu y',\mu^2 s'), \;\; B_{\mu}(y',s') = B(\mu y',\mu^2 s') \\
&y' = e^{-s/2}y, \;\; s' = -e^{-s},
\end{aligned} \tag{1.51}
$$

where $\mu > 0$ is a constant.

The condition (1.48), however, is a scaling invariant. In fact,

$$
\left\langle c(|y|^2)+1, \zeta(dy,s) \right\rangle < \delta \tag{1.52}
$$

for some $s \in \mathbf{R}$ means

$$
\left\langle c\left((-s')^{-1}|y'|^2\right), A(dy,s') \right\rangle < \delta
$$

for some $s' < 0$, while

$$
\left\langle c\left((-s')^{-1}|y'|^2\right), A_{\mu}(dy,s') \right\rangle < \delta \tag{1.53}
$$

is equivalent to

$$
\left\langle c\left((-\mu^2 s')^{-1}|y'|^2\right), A(dy,\mu^2 s') \right\rangle < \delta. \tag{1.54}
$$

It is thus impossible to find $\mu > 0$ satisfying

$$
\left\langle c(|y|^2)+1, \zeta^{\mu}(dy,0) \right\rangle < \delta
$$

for the general case. All the difficulties lie already in the fact that (1.50) is defined only for $s' < 0$. In the following section, we complete the proof of mass quantization.

1.1.7. Mass Quantization

Taking an alternative approach to the proof of Theorem 1.2, we recall (1.35) in the form of

$$
\left| \frac{d}{dt}\int_{\Omega} u(\cdot,t)\varphi \right| \leq C\,\|\varphi\|_{C^2(\overline{\Omega})} \tag{1.55}
$$

with a constant $C = C_{\lambda} > 0$ independent of $\varphi \in C^2(\overline{\Omega})$ and that inequality (1.55) is derived from (1.33). Here, we take $\varphi_{x_0,R}$, the smooth cut-off function with the support radius $2R > 0$, and apply (1.33) to $\varphi = |x-x_0|^2\,\varphi_{x_0,R}$. This property results in, see (5.32) of,[304]

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega} |x-x_0|^2 \varphi_{x_0,R}\cdot u(\cdot,t) \\
&\quad \leq 4A_R(t) + C\left\{ \frac{1}{R^2}\int_{\Omega} |x-x_0|^2 \varphi_{x_0,R}\cdot u(\cdot,t) \right\}^{1/2} \\
&\qquad + C\int_{\Omega} (\varphi_{x_0,2R}-\varphi_{x_0,R})\cdot u(\cdot,t)
\end{aligned} \tag{1.56}
$$

with a constant $C > 0$ independent of $0 < R \leq 1$, where

$$
A_R(t) = \int_{\Omega} \varphi_{x_0,R}\cdot u(\cdot,t)\cdot \left\{ 1 - \frac{1}{m_*(x_0)}\int_{\Omega} \varphi_{x_0,R}\cdot u(\cdot,t) \right\}.
$$

In case $x_0 \in \partial\Omega$, actually, we modify $|x-x_0|^2$ by a suitable function $M = M(x)$ satisfying $\frac{\partial M}{\partial \nu} = 0$ on $\partial\Omega$.
From (1.11), the right-hand side of (1.56) converges to

$$4\left\{m(x_0) + \int_\Omega \varphi_{x_0,R} \cdot f\right\}\left\{1 - \frac{1}{m_*(x_0)}\left(m(x_0) + \int_\Omega \varphi_{x_0,R} \cdot f\right)\right\}$$
$$+C\left\{\frac{1}{R^2}\int_\Omega |x-x_0|^2 \varphi_{x_0,R} \cdot f\right\}^{1/2} + C\int_\Omega (\varphi_{x_0,2R} - \varphi_{x_0,R}) \cdot f$$

as $t \uparrow T$ for $0 < R \ll 1$, which is estimated from above by

$$4\left\{m(x_0) + \int_\Omega \varphi_{x_0,R} \cdot f\right\}\left\{1 - \frac{1}{m_*(x_0)}\left(m(x_0) + \int_\Omega \varphi_{x_0,R} \cdot f\right)\right\}$$
$$+2C\left\{\int_\Omega \varphi_{x_0,R} \cdot f\right\}^{1/2} + C\int_\Omega \varphi_{x_0,2R} \cdot f$$
$$= 4m(x_0)\left\{1 - \frac{m(x_0)}{m_*(x_0)}\right\} + o(1)$$

as $R \downarrow 0$. In case $m(x_0) > m_*(x_0)$, or, equivalently $\hat{m}(x_0) > 8\pi$, therefore, it holds that

$$\frac{d}{dt}\int_\Omega |x-x_0|^2 \varphi_{x_0,R} \cdot u(\cdot,t) \le -a$$

for $0 < T-t \ll 1$ and $0 < R \ll 1$, where $a > 0$ is a constant. This inequality implies

$$\int_\Omega |x-x_0|^2 \varphi_{x_0,R} \cdot u(\cdot,t) \le -a(t-\tilde{t}) + \int_\Omega |x-x_0|^2 \varphi_{x_0,R} \cdot u(\cdot,\tilde{t})$$
$$\le -a(t-\tilde{t}) + \int_\Omega |x-x_0|^2 \varphi_{x_0,\tilde{R}} \cdot u(\cdot,\tilde{t})$$

for $0 < T-t \le T-\tilde{t} \ll 1$ and $0 < R \le \tilde{R} \ll 1$. Putting $R = bR(t)$ and $\tilde{R} = bR(\tilde{t})$ with $b > 0$ and $R(t) = (T-t)^{1/2}$, and thus we obtain

$$\int_\Omega \left|\frac{x-x_0}{R(t)}\right|^2 \varphi_{x_0,bR(t)} \cdot u(\cdot,t)$$
$$\le -a \cdot \frac{t-\tilde{t}}{T-t} + \frac{T-\tilde{t}}{T-t}\int_\Omega \left|\frac{x-x_0}{R(\tilde{t})}\right|^2 \varphi_{x_0,bR(\tilde{t})} u(\cdot,\tilde{t}) \tag{1.57}$$

for $0 < T-t \le T-\tilde{t} \ll 1$.
Given $t_k \uparrow T$, we take $\{s'_k\} \subset \{s_k\}$ and $\zeta(dy,s)$ satisfying (1.41) in

$$C_*(-\infty,+\infty;\mathscr{M}(\mathbf{R}^2)),$$

where $s_k = -\log(T-t_k)$. We take also $s \in \mathbf{R}$ and define t'_k and $\tilde{t}_k$ by

$$s + s'_k = -\log(T-t'_k)$$
$$t'_k - \tilde{t}_k = T - t'_k.$$

It holds that $T-\tilde{t}_k = 2(T-t'_k)$, and, therefore, inequality (1.57) for $t = t'_k$ and $\tilde{t} = \tilde{t}_k$ reads;

$$\int_\Omega \left|\frac{x-x_0}{R(t'_k)}\right|^2 \varphi_{x_0,bR(t'_k)} \cdot u(\cdot,t'_k)$$
$$\leq -a + 2\int_\Omega \left|\frac{x-x_0}{R(\tilde{t}_k)}\right|^2 \varphi_{x_0,bR(\tilde{t}_k)} \cdot u(\cdot,\tilde{t}_k),$$

and we have

$$\tilde{s} + s'_k \equiv -\log(T-\tilde{t}_k) = s + s'_k - \log 2.$$

We thus obtain

$$\begin{aligned}\left\langle |y|^2\varphi_{0,b}, \zeta(dy,s)\right\rangle &\leq -a + 2\left\langle |y|^2\varphi_{0,b}, \zeta(dy,\tilde{s})\right\rangle \\ &= -a + 2\left\langle |y|^2\varphi_{0,b}, \zeta(dy,s-\log 2)\right\rangle \end{aligned} \tag{1.58}$$

by sending $k \to \infty$.
From the formation of sub-collapse in the rescaled solution, however, the singular part of $\zeta(dy, s-\log 2)$ is composed of a finite sum of delta functions, see,[281] Chapter 14 of,[304] and §1.1.8. It holds, therefore, that

$$\left\langle |y|^2\varphi_{0,b}, \zeta(dy,s-\log 2)\right\rangle = o(1) \tag{1.59}$$

as $b \downarrow 0$. The right-hand side of (1.58) is thus negative for $0 < b \ll 1$, a contradiction. The proof of Theorem 1.2 is complete. □

1.1.8. Blowup Rate

The scaling invariance (1.53)-(1.54) of the rescaled second moment, however, induces an important property concerning the blowup rate of the solution. Described in §1.1.6, the formal blowup rate is obtained by the ODE part of (1.45),

$$u_t = \Delta u - \nabla u \cdot \nabla v + u^2,$$

that is

$$\dot{m} = m^2.$$

Since the blowup solution to this ODE is given by $m(t) = (T-t)^{-1}$, we say that the rate of the blowup solution to (1.8) is type (I) if

$$\limsup_{t\uparrow T}(T-t)\|u(\cdot,t)\|_\infty < +\infty.$$

Then, the classification of the blowup point is done in accordance with the backward self-similar transformation (1.36)-(1.37). We say that $x_0 \in \mathscr{S}$ is of type (I) if

$$\limsup_{t\uparrow T} R(t)^2 \|u(\cdot,t)\|_{L^\infty(B(x_0,bR(t)\cap\Omega)} < +\infty$$

for any $b>0$, where $R(t)=(T-t)^{1/2}$, and we say that $x_0\in\mathscr{S}$ is of type (II) if it is not of type (I), that is if there is $t_k\uparrow T$ and $b>0$ such that

$$\lim_{k\to\infty} R(t_k)^2\|u(\cdot,t_k)\|_{L^\infty(B(x_0,bR(t_k))\cap\Omega)}=+\infty.$$

Then, we obtain the following theorem.[215,276]

Theorem 1.5. *Every $x_0\in\mathscr{S}$ is of type (II). More strongly, we have*

$$\lim_{t\uparrow T} R(t)^2\,\|u(\cdot,t)\|_{L^\infty(B(x_0,bR(t))\cap\Omega)}=+\infty \tag{1.60}$$

for any $b>0$ and

$$z(y,s+s')dy\rightharpoonup m_*(x_0)\delta_0(dy) \tag{1.61}$$

in $C_*(-\infty,+\infty;\mathscr{M}(\mathbf{R}^2))$ as $s'\uparrow+\infty$, where $R(t)=(T-t)^{1/2}$ and $z=z(y,s)$ denotes the backward self-similar transformation of $u=u(x,t)$ defined by (1.36)-(1.37).

Relation (1.61) is called the formation of *sub-collapse*. It says that the total blowup mechanism is enclosed in the infinitesimally small parabolic region, which we call the *hyper-parabola*. We see that type (II) blowup rate indicated by (1.60) is obtained by this (1.61). For the proof, we pick up a preliminary result,[281,304] which has been already used in the proof of (1.59). It is, however, not so sharp as (1.61); the possibilitis of collapse collision and the residual term are left. More precisely, the $*$-weakly continuous measure $\zeta(dy,s)$, generated by $s_k=-\log(T-t_k)$ through (1.41), takes the form

$$\zeta(dy,s)=\sum_{y_0\in\mathscr{B}_s}8\pi\delta_{y_0}(dy)+g(y,s)dy \tag{1.62}$$

for each $s\in\mathbf{R}$. Here, $\mathscr{B}_s$ is a finite set and $0\le g=g(\cdot,s)\in L^1(\mathbf{R}^2)\cap C(\mathbf{R}^2\setminus\mathscr{B}_s)$. Equality (1.62) is actually a similarity to the quantized blowup mechanism arising in the blowup solution in infinite time to the pre-scaled system (1.8). Thus, we perform a hierarchical argument[280] for the proof of (1.62). The proof is, therefore, quite similar to that of the prescaled case of which fundamenta idea is illustrated in §1.1.9.

To improve (1.62), we show, next, the convergence of the global second moment of the rescaled solution. To this end, we note

$$\frac{d}{dt}\int_\Omega |x-x_0|^2\,\varphi_{x_0,R}\cdot u(\cdot,t)\ge -C$$

obtained similarly to (1.56) in spite of $\hat{m}(x_0)=8\pi$, where $C>0$ is a constant independent of $0<R\le 1$. This time, we have

$$\int_\Omega |x-x_0|^2\,\varphi_{x_0,R}\cdot u(\cdot,t)\le C(T-t)+\int_\Omega |x-x_0|^2\,\varphi_{x_0,R}\cdot f$$

by (1.11), and, therefore,

$$\begin{aligned}\frac{1}{R(t)^2}&\int_\Omega |x-x_0|^2\,\varphi_{x_0,bR(t)}\cdot u(\cdot,t)\\ &\le C+\frac{1}{R(t)^2}\int_\Omega |x-x_0|^2\,\varphi_{x_0,bR(t)}\cdot f\\ &\le C+4b^2\left\langle\varphi_{x_0,bR(t)},f\right\rangle.\end{aligned} \tag{1.63}$$

by putting $R = bR(t)$.
Given $t_k \uparrow T$, we take $\{s'_k\} \subset \{s_k\}$ satisfying (1.41) in

$$C_*(-\infty, +\infty; \mathscr{M}(\mathbf{R}^2))$$

as before, where $s_k = -\log(T - t_k)$. It follows that

$$\left\langle |y|^2, \zeta(dy, s) \right\rangle \leq C \qquad -\infty < s < \infty \tag{1.64}$$

from $b \uparrow +\infty$.
Putting

$$I(s) = \left\langle |y|^2, \zeta(dy, s) \right\rangle,$$

now we obtain

$$\frac{dI}{ds} = 4\hat{m}(x_0) - \frac{\hat{m}(x_0)^2}{2\pi} + I = I \qquad \text{a.e. } s \in \mathbf{R} \tag{1.65}$$

by (1.42) and $\hat{m}(x_0) = \zeta(\mathbf{R}^2, s) = 8\pi$. Then, it follows that

$$\left\langle |y|^2, \zeta(dy, s) \right\rangle = 0, \qquad -\infty < s < \infty \tag{1.66}$$

from (1.64)-(1.65). Combining (1.62) and (1.66), we obtain

$$\zeta(dy, s) = 8\pi\delta_0(dy), \tag{1.67}$$

and, therefore, (1.61). This convergence implies (1.60) and the proof of Theorem 1.5 is complete. □

Free Energy Transmission

Formation of sub-collapse, (1.61), suggests the other scaling $r(t) \ll R(t) = (T - t)^{1/2}$ to describe real blowup envelope for type (II) blowup point. Herrero-Velázquez' solution[142] is a radially symmetric solution to (1.8) satisfying

$$u(x,t) = \frac{\overline{u}(x/r(t))}{r(t)^2} \left[1 + o(1) + O\left(\frac{e^{-\sqrt{2}|\log(T-t)|^{1/2}}}{|x|^2} \cdot 1_{x \geq r(t)} \right) \right]$$

as $t \uparrow T = T_{\max} < +\infty$ uniformly in $x \in B(0, bR(t))$ for any $b > 0$, where

$$\begin{aligned} r(t) &= (T-t)^{1/2} e^{-\sqrt{2}|\log(T-t)|^{1/2}} \left|\log(T-t)\right|^{\frac{1}{4}\left(|\log(T-t)|^{-1/2} - 1\right)} \\ &\cdot \ (1 + o(1)) \ll R(t) \end{aligned}$$

and

$$\overline{u}(y) = \frac{8}{\left(1 + |y|^2\right)^2}$$

are the blowup rate and an *entire stationary state* satisfying

$$-\Delta \log \overline{u} = \overline{u} \qquad \text{in } \mathbf{R}^2,$$

respectively, see (1.45). This solution obeys an interesting feature, which we call the *emergence*. In more details, the local free energy diverges to $+\infty$ around it, that is

$$\lim_{t\uparrow T}\mathscr{F}_{x_0,br(t)}(u(\cdot,t))=+\infty \tag{1.68}$$

for any $b>0$, where

$$\mathscr{F}_{x_0,R}(u)=\int_\Omega u(\log u-1)\varphi_{x_0,R}$$
$$-\frac{1}{2}\iint_{\Omega\times\Omega}G(x,x')(u\cdot\varphi_{x_0,R})(x)(u\cdot\varphi_{x_0,R})(x')dxdx'.$$

This closed system, thus, realizes the space and time localization of the free energy. Since it holds also that

$$\limsup_{t\uparrow T}\int_\Omega \varphi_{x_0,bR(t)}u(\cdot,t)<m_*(x_0)$$

for any $b>0$, we can summarize that "mass and entropy are exchanged at the wedge of the blowup envelope, creating a clean, quantized self".[304] Such a profile is expected for a wide class of solutions with $r(t)=u(x_0,t)^{-1/2}$.

1.1.9. Blowup in Infinite Time

The following are open problems concerning the blowup mechanism for (1.8) with $n=2$.

(1) classification of the blowup envelope $r(t)\downarrow 0$ for the general type (II) blowup point, that is

$$\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,br(t)))}<m_*(x_0),\quad b>0$$
$$\lim_{b\uparrow+\infty}\lim_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,br(t)))}=m_*(x_0)$$

and the profile of the emergence, (1.68) for any $b>0$.

(2) to prescribe the blowup set $\mathscr{S}\subset\overline{\Omega}$ composed of arbitrary finite points.

(3) to exclude the boundary blowup point when the boundary condition is replaced by

$$\frac{\partial u}{\partial \nu}-u\frac{\partial v}{\partial \nu}=v=0 \qquad \text{on } \partial\Omega\times(0,T). \tag{1.69}$$

(4) global in time existence of the weak solution similarly to the harmonic heat flow, see §1.1.11.

Blowup in infinte time is another issue. If the principle of the quantized blowup mechanism is valid even in infinite time, what is expected is that the blowup in infinite time occurs only for $\lambda=\|u_0\|_1\in 4\pi\mathbf{N}$. The following theorem is to be noticed first.

Theorem 1.6. *Inequality (1.14) of Theorem 1.2 is improved by*

$$2\sharp(\mathscr{S}\cap\Omega)+\sharp(\mathscr{S}\cap\partial\Omega)<\|u_0\|_1/(4\pi). \tag{1.70}$$

Proof: From the parabolic and the elliptic local regularities, $u = u(x,t)$ and $v = v(x,t)$ have smooth extensions to $Q \equiv \overline{\Omega} \times [0,T] \setminus \mathscr{S} \times \{T\}$, and it holds that

$$u_t = \nabla \cdot (\nabla u - u \nabla v) \qquad \text{in } Q,$$

where $T = T_{\max} < +\infty$. Therefore, if $u(x^*, T) = 0$ for some $x^* \in \overline{\Omega} \setminus \mathscr{S}$, then we obtain $u \equiv 0$ on Q by the maximum principle or unique continuation theorem, a contraction. This contradiction means $f(x) > 0$ for $x \in \overline{\Omega} \setminus \mathscr{S}$ in (1.11), and in particular,

$$\|u_0\|_1 = \sum_{x_0 \in \mathscr{S}} m_*(x_0) + \|f\|_1 > \sum_{x_0 \in \mathscr{S}} m_*(x_0).$$

Hence (1.70) follows. □

Theorem 1.6 guarantees, in particular, $T_{\max} = +\infty$ if $\lambda \equiv \|u_0\|_1 = 4\pi$. If $\limsup_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty < +\infty$, then the orbit $\mathscr{O} = \{u(\cdot,t)\}_{t \geq 0}$ is compact in $C(\overline{\Omega})$, and then the *ω-limit set* of $\mathscr{O}$ is defined by

$$\omega(u_0) = \{u_\infty \in C(\overline{\Omega}) \mid \text{there is } t_k \uparrow +\infty \text{ such that } u(\cdot,t_k) \to u_\infty \text{ in } C(\overline{\Omega})\}.$$

Under the presense of the Lyapunov function $\mathscr{F} = \mathscr{F}(u)$, this ω-limit set is non-void, compact, connected, and is contained in E, the total set of stationary solutions, see[138] and also §1.2.7. If this boundedness of the orbit is not the case then it holds that $\limsup_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty = +\infty$, and hence there is $t_k \uparrow +\infty$ such that

$$\|u(\cdot,t_k)\|_\infty \to +\infty. \tag{1.71}$$

Formation of the Quantized Collapse

The following theorem is concerned with the general case of $T_{\max} = +\infty$, so that admits the possibility of $n(t) = 0$.

Theorem 1.7 ([280]). *If $T = T_{\max} = +\infty$ holds to (1.8), any $t_k \uparrow +\infty$ admits $\left\{t_k'\right\} \subset \{t_k\}$ such that*

$$u(x, t + t_k')dx \rightharpoonup \mu(dx,t) \tag{1.72}$$

in $C_*\left(-\infty, +\infty; \mathscr{M}(\overline{\Omega})\right)$, where $\mu(dx,t)$ is a weak solution to (1.8). This term means that there is $0 \leq \nu = \nu(\cdot,t) \in L_*^\infty(-\infty,+\infty;\mathscr{E}')$ satisfying

$$\frac{d}{dt}\langle \varphi, \mu(dx,t)\rangle = \langle \Delta\varphi, \mu(dx,t)\rangle + \frac{1}{2}\left\langle \rho_\varphi, \nu(\cdot,t)\right\rangle_{\mathscr{E},\mathscr{E}'} \quad \text{a.e. } t \in \mathbf{R}$$

for $\varphi = \varphi(x) \in C^2(\overline{\Omega})$ with $\left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$, where

$$\rho_\varphi(x,x') = \nabla\varphi(x) \cdot \nabla_x G(x,x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x,x')$$

$$\mathscr{E} = \left\{\rho_\varphi \mid \varphi \in C^2(\overline{\Omega}),\ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\right\}$$

$$\nu(\cdot,t)|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu(dxdx',t) \qquad \text{a.e. } t.$$

Furthermore, the singular part of $\mu(dx,t)$, denoted by $\mu_s(dx,t)$, is composed of a finite sum of delta functions with the quantized mass,

$$\mu_s(dx,t) = \sum_{i=1}^{n(t)} m_*(x_i(t))\delta_{x_i(t)}(dx). \tag{1.73}$$

Since the regular part $f = f(x,t)$ of $\mu(dx,t)$ satisfies a parabolic equation in the relatively open set

$$\bigcup_{t\in\mathbf{R}} (\overline{\Omega} \setminus \{x_i(t) \mid 1 \le i \le n(t)\}) \times \{t\}$$

of $\overline{\Omega} \times \mathbf{R}$, we obtain $\mu(dx,t) = \mu_s(dx,t)$ if

$$\lambda = \sum_{i=1}^{n(0)} m_*(x_i(0)) \tag{1.74}$$

is the case, similarly to the proof of Theorem 1.6. This fact, of course, is not sufficient to infer $T_{\max} = +\infty$ for $\lambda \notin 4\pi\mathbf{N}$.
The estimate of sub-collapse mass from above in (1.62), formed in infinite time to the rescaled solution, plays a key role in the proof of Theorem 1.5, that is type (II) blowup rate of the pre-scaled solution blowing-up in finite time. This estimate is done similarly to the pre-scaled case (1.73), in the flavor of the description at the end of §1.1.5; the "post-blowup continuation" implies this estimate as the localization of the blowup criterion. To estimate the collapse mass from below, on the other hand, we have to discretize in time of the global in time existence criterion of the solution, Theorem 1.3, besides the localization in space. This proof is achieved by the parabolic regularity or the smothing effect and the reduction to radially symmetric case by the Schwarz symmetrization. We call this part the *concentration lemma*, see Chapter 12 of.[304]

Collapse Dynamics

If (1.71) occurs to $\lambda = 4\pi$, it thus holds that

$$\mu(dx,t) = 4\pi\delta_{x(t)}(dx)$$

with $t \in \mathbf{R} \mapsto x(t) \in \partial\Omega$ locally absolutely continuous. The movement of collapse, on the other hand, is controlled by the Robin function from the method of second moment,[233] and thus, we have the following theorem.

Theorem 1.8. *If $\lambda \equiv \|u_0\|_1 = 4\pi$, then it holds that $T_{\max} = +\infty$. If there is $t_k \uparrow +\infty$ satisfying (1.71), then we have $\{t'_k\} \subset \{t_k\}$ satisfying (1.72) with $\mu(dx,t) = 4\pi\delta_{x(t)}(dx)$. Here,*

$$t \in \mathbf{R} \mapsto x(t) \in \partial\Omega$$

is locally absolutely continuous, and is subject to

$$\frac{dx}{dt} = 2\pi\nabla_\tau R(x)$$

for a.e. $t \in \mathbf{R}$, where τ is the tangential unit vector on $\partial\Omega$, $\nabla_\tau = \tau(\tau \cdot \nabla)$, and $R(x) = K(x,x')$ is the Robin function defined from the regular part $K(x,x')$ of the Green's function defined by (1.28).

Since

$$\frac{d}{dt} R(x(t)) \geq 0,$$

we have $\lim_{t\to\pm\infty} x(t) = x_\infty^\pm \in \partial\Omega$, where x_∞^- and x_∞^+ are a local minimizer and a local maximizer of $R|_{\partial\Omega}$, respectively. In other words, this collapse $4\pi\delta_{x(t)}(dx)$ formed in infinite time, appears first at x_∞^- on $\partial\Omega$, and then moves to x_∞^+ along $\partial\Omega$.
Similarly, we obtain the following theorem.[233]

Theorem 1.9. *Let the boundary condition be replaced by (1.69) in (1.8), and assume that $\lambda = \|u_0\|_1 = 8\pi$, $T_{\max} = +\infty$, and there is no stationary solution $v = v(x)$ to*

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{1.75}$$

Then, any $t_k \uparrow +\infty$ admits $\{t_k'\} \subset \{t_k\}$ such that

$$\begin{aligned}
& u(x,t+t_k')dx \rightharpoonup 8\pi\delta_{x(t)}(dx) && \text{in } C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega})) \\
& t \in \mathbf{R} \mapsto x(t) \in \Omega && \text{absolutely continuous} \\
& \liminf_{t\uparrow+\infty} dist(x(t),\partial\Omega) > 0 && \\
& \frac{dx}{dt} = 4\pi\nabla R(x(t)) && \text{a.e. } t \in \mathbf{R},
\end{aligned} \tag{1.76}$$

where

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'| \right]_{x'=x}$$

with $G = G(x,x')$ standing for the Green's function to $-\Delta$ with $\cdot|_{\partial\Omega} = 0$.

The ordinary differential equation descibed by the last equality of (1.76) is a conjugate of the vortex equation derived from the Euler equation, see §2.2.1,

$$\frac{dx}{dt} = 4\pi\nabla^\perp R(x(t)).$$

Global in time behavior of the radially symmetric solution, finally, is classified as follows.[233]

Theorem 1.10. *If $\Omega = B(0,R) = \{x \in \mathbf{R}^2 \mid |x| < R\}$ is a disc, $u_0 = u_0(|x|)$ is radially symmetric, and $\lambda = \|u_0\|_1 > 8\pi$, then the blowup in infinite time does not occur to (1.8). Thus, we have either $T = T_{\max} < +\infty$ or $T_{\max} = +\infty$ and $\limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty < +\infty$. If the boundary condition is replaced by (1.69), then we have always $T = T_{\max} < +\infty$ for $\lambda > 8\pi$.*

If the blowup in infinite time does not occur, then the ω-limit set of the orbit global in time is non-empty, connected, and compact in $C(\overline{\Omega})$, and is also contained in the set of stationary solutions, see §1.2.7. The total set of stationary solutions to (1.8) and that with the boundary condition replaced by (1.69), however, are rather different. In the former case, first, we have always the constant stationary solution $u = \lambda/|\Omega|$. If $\Omega = B(0,R)$, furthermore, many non-radially symmetric solutions arise. There are also non-constant radially symmetric stationary solution for $\lambda > 8\pi$. If $0 < \lambda < 8\pi$, then any radially symmetric solution is constant. In the latter case, the stationary problem is described by (1.75) in v-component. The solution is always unique, provided that $0 < \lambda < 8\pi$. If $\Omega = B(0,R)$, we obtain the singular limit of $v = v(x)$ as $\lambda \uparrow 8\pi$, i.e.,

$$v_\infty(x) = 4\log\frac{R}{|x|}. \tag{1.77}$$

See[277,304] and §2.3.4 for more details.
Coming back to Theorem 1.10 concerning radially symmetric non-stationary solutions, if $\lambda < 8\pi$ then any solution exists globally in time, see,[277] and, therefore, converges to the unique stationary state in both cases of the boundary condition. If $\lambda \geq 8\pi$ and the boundary condition is (1.69), there is no stationary solution to (1.75), and, actually, we have the blowup in finite time in the case of $\lambda > 8\pi$. What is expected to this boundary condition with $\lambda = 8\pi$ is that the v-component of the solution converges to the singular stationary state (1.77) in infinite time. Concerning (1.8), there is a chance that the solution exists globally in time and converges to the constant $\lambda/|\Omega|$ or the other radially symmetric stationary solution even if $\lambda \geq 8\pi$.

1.1.10. Time Relaxization

The full system of chemotaxis is provided with the time relaxization. A typical form is given by

$$\begin{aligned}
& u_t = \nabla\cdot(\nabla u - u\nabla v) \\
& \tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \qquad && \text{in } \Omega\times(0,T) \\
& \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega\times(0,T) \\
& \int_\Omega v = 0, && 0 < t < T,
\end{aligned} \tag{1.78}$$

with $\tau > 0$ in the left-hand side of the second equation standing for the relaxization time which describes a chemical process of the formation of the field v. The fundamental structure is the existence of the Lagrangian,

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \int_\Omega vu \tag{1.79}$$

defined for

$$\int_\Omega v = 0,$$

and (1.78) is equivalent to

$$\begin{aligned} & u_t = \nabla\cdot(u\nabla L_u(u,v)) \\ & \tau v_t = -L_v(u,v) && \text{in } \Omega\times(0,T) \\ & u\frac{\partial}{\partial\nu}L_u(u,v) = \frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\ & \int_\Omega v = 0, && 0<t<T. \end{aligned}$$

Thus it holds that

$$\frac{d}{dt}L(u,v) = -\int_\Omega u\,|\nabla L_u|^2 + \tau^{-1}v_t^2\;dx \leq 0$$

for the solution $(u,v) = (u(\cdot,t),v(\cdot,t))$. This formulation to (1.78) is assocaited with *dual variation*, and, consequently, any linearly stable stationary solution is dynamically stable, see §1.2.

Well-posedness locally in time and the standard blowup criterion are proven also to this system. More precisely, we have always $0 < T_{\max} \leq +\infty$. If $T = T_{\max} < +\infty$, furthermore, then $\lim_{t\uparrow T}\|u(\cdot,t)\|_\infty = +\infty$ and the blowup set $\mathscr{S}$ of u, defined by (1.12), is not empty. The blowup criterion is also valid, and similarly to Theorem 1.3, $\lambda = \|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$ for the general case, while $\lambda < 8\pi$ implies $T_{\max} = +\infty$ in the radially symmetric case.[24,108,211]

The local blowup criterion, or ε-regularity, also holds and we obtain

$$\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq m_*(x_0)$$

for each $x_0 \in \mathscr{S}$, where $R > 0$ is arbitrary.[212] The property of bounded variation in time of the local L^1-norm derived from (1.35) in the case of $\tau = 0$, however, is not known, and, consequently, the finiteness of the blowup points is open to (1.78). The threshold blowup, or even any blowup criterion in finite time such as Theorem 1.4 is also not known to (1.78).

Non-Local Parabolic Equation

Replacing u_t by εu_t in (1.78) and making $\varepsilon \downarrow 0$, we obtain (1.24) which induces another simplified system of chemotaxis,

$$\begin{aligned} & \tau v_t = \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) && \text{in } \Omega\times(0,T) \\ & \frac{\partial v}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\ & \int_\Omega v = 0, && 0<t<T \end{aligned} \tag{1.80}$$

formulated by Wolansky.[341,342] This problem is provided with the Lyapunov function

$$\mathscr{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1), \qquad \int_\Omega v = 0,$$

and it holds that

$$\frac{d}{dt}J_\lambda(v) = -\tau^{-1}\|v_t\|_2^2 \leq 0.$$

Not so much is known to (1.80) also, but the following theorems show the *dis-quantized blowup mechanism* to its relative,

$$\tau v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v} \qquad \text{in } \Omega \times (0,T)$$
$$v = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.81}$$

Theorem 1.11 ([341]). *If* $\Omega = B(0,R)$*,* $v_0 = v_0(|x|)$*, and* $\lambda \geq 8\pi$*, then*

$$u \equiv \frac{\lambda e^v}{\int_\Omega e^v} \rightharpoonup \lambda\delta_0 \tag{1.82}$$

as $t \uparrow T = T_{\max} \in (0,+\infty]$ *in* $C(\overline{\Omega})$*, where* $v = v(\cdot,t)$ *is the solution to (1.81) with* $v(\cdot,0) = v_0$*.*

Theorem 1.12 ([163]). *If* $\lambda > 8\pi$ *in the previous theorem, it holds that* $T = T_{\max} < +\infty$*, and, therefore, we have the formation of collapse (1.82) in finite time with the dis-quantized mass* $\lambda > 8\pi$*.*

The threshold case $\lambda = 8\pi$ is open, but the blowup in infinite time is expected.

Non-Local ODE

Spatially homogeneous part of (1.80) or (1.81) is formulated by the *non-local ODE*. Adopting the normalization $\tau = 1$, $\lambda = |\Omega|$, we obtain

$$v_t = \frac{|\Omega|e^v}{\int_\Omega e^v} - 1 \qquad \text{in } \Omega \times (0,T) \tag{1.83}$$

or

$$v_t = \frac{|\Omega|e^v}{\int_\Omega e^v} \qquad \text{in } \Omega \times (0,T) \tag{1.84}$$

Even these equations are not so trivial.
First, equation (1.83) is reduced to (1.84) by $\tilde{v} = v + t$. Next, in (1.84) it holds that

$$\frac{1}{|\Omega|}\frac{d}{dt}\int_\Omega v = 1$$

and hence

$$\frac{1}{|\Omega|}\int_\Omega v = t + \frac{1}{|\Omega|}\int_\Omega v_0. \tag{1.85}$$

We have, furthermore,

$$-(e^{-v})_t = a(t) \equiv \frac{|\Omega|}{\int_\Omega e^v},$$

and

$$e^{-v_0(x)} - e^{-v(x,t)} = A(t) \equiv \int_0^t a(t')dt', \tag{1.86}$$

where $v_0 = v(\cdot, 0)$ which implies that

$$e^{-v_0(x_1)} - e^{-v(x_1,t)} = e^{-v_0(x_2)} - e^{-v(x_2,t)},$$

and, hence, the *order preserving property*

$$v_0(x_1) \le v_0(x_2) \quad \Rightarrow \quad v(x_1,t) \le v(x_2,t).$$

In particular, it holds that

$$\|v_0\|_\infty = v_0(x_*) \quad \Rightarrow \quad \|v(\cdot,t)\|_\infty = v(x_*,t).$$

Since $v_t > 0$, we have

$$\lim_{t\uparrow T} v(x,t) = v(x,T) \in (-\infty, +\infty]$$

for each $x \in \overline{\Omega}$, and, therefore,

$$\lim_{t\uparrow T} \|v(\cdot,t)\|_\infty = +\infty \quad \Rightarrow \quad v(x_*,T) = +\infty.$$

It follows that

$$\begin{aligned} e^{-v_0(x_*)} &= e^{-\|v_0\|_\infty} = A(T) \\ e^{-v_0(x)} - e^{-v(x,T)} &= A(T) \end{aligned}$$

from (1.86), so that

$$v(x,T) = -\log\{e^{-v_0(x)} - e^{-\|v_0\|_\infty}\}.$$

Pluging this relation into (1.85), we obtain

$$-\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0} - e^{-\|v_0\|_\infty}\} = T + \frac{1}{|\Omega|}\int_\Omega v_0,$$

and hence the following theorem.

Theorem 1.13. *The solution $v = v(x,t)$ to the non-local ODE (1.84) blows-up at the blowup time*

$$T = -\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0} - e^{-\|v_0\|_\infty}\} - \frac{1}{|\Omega|}\int_\Omega v_0.$$

We note that there is a case of $T = +\infty$ while it holds always that

$$T > -\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0}\} - \frac{1}{|\Omega|}\int_\Omega v_0 = 0.$$

Cancer Model

Several mathematical models have been proposed to describe the movement of living things attracted by non-diffusive chemical factors using re-inforced random walk which results in parabolic-ODE systems in the limit state, that is

$$\begin{aligned} p_t &= \nabla \cdot (D\nabla p - p\chi(w)\nabla w) \\ w_t &= g(p,w), \end{aligned} \tag{1.87}$$

see.[247] Here, p and w are due to the conditional probability of the decision of the walkers and the density of the control species, respectively, $D > 0$ the diffusion constant, χ the chemotactic sensitivity, and g the chemical growth rate.

Angiogenesis is the formation of blood vessels from a pre-existing vasculature. It is a process whereby capillary sprouts are formed in response to externally supplied stimuli and provides with a drastic stage to the tumor growth. A parabolic-ODE system modelling tumor-induced angiogenesis is proposed in this connection,[8] using the endotherial-cell density per unit area n, the TAF (tumor angiogenic factors) concentration f, and the matrix macromolecucule fibronectin concentration c, that is,

$$\begin{aligned} n_t &= D\Delta n - \nabla \cdot (\chi(c)n\nabla c) - \rho_0 \nabla \cdot (n\nabla f) \\ f_t &= \beta n - \mu n f \\ c_t &= -\gamma n c \qquad \text{in } \Omega \times (0,T), \end{aligned} \tag{1.88}$$

where

$$\chi(c) = \frac{\chi_0}{1+\alpha c}$$

denotes the chemotactic sensitivity and D, ρ_0, β, μ, γ, χ_0, α are positive constants. System (1.87) is formulated as an evolution equation with strong dissipation.[173,179,349] There is also an approach from the comparison theorem.[104] Here, we take the third approach using the calculus of variation.

We formulate the problem as the parabolic-ODE system

$$\begin{aligned} q_t &= \nabla \cdot (\nabla q - q\nabla \varphi(v)) \\ v_t &= q \qquad \text{in } \Omega \times (0,T) \end{aligned} \tag{1.89}$$

with

$$\begin{aligned} &\frac{\partial q}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\ &q|_{t=0} = q_0, \qquad v|_{t=0} = v_0 \qquad \text{on } \overline{\Omega}, \end{aligned} \tag{1.90}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outer unit normal vector, $q_0 > 0$ and v_0 are smooth function on $\overline{\Omega}$, and $\varphi : \mathbf{R} \to \mathbf{R}$ is a smooth function.[310] We impose the compatibility condition

$$\frac{\partial v_0}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

which guarantees the null flux boundary condition

$$\frac{\partial q}{\partial \nu} - q\frac{\partial \varphi(v)}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).$$

Although the form of (1.89) is restrictive, some important cases of (1.87) are reduced to it. First, if

$$g(p,w) = (p-\mu)w, \qquad w > 0$$

then we obtain (1.89) by

$$\begin{aligned} &v = \log w, \;\; q = p - \mu \\ &\varphi(v) = A(e^v), \;\; A' = \chi. \end{aligned}$$

Next, if

$$g(p,w) = p(\mu - w), \qquad w < \mu$$

then (1.89) follows from

$$\begin{aligned} &v = -\log(\mu - w), \;\; q = p \\ &\varphi(v) = A(\mu - e^{-v}), \;\; A' = \chi. \end{aligned}$$

Finally, if

$$g(p,w) = -pw, \qquad w > 0,$$

then (1.89) holds for

$$\begin{aligned} &v = -\log w, \;\; q = p \\ &\varphi(v) = A(e^{-v}), \;\; A' = \chi. \end{aligned}$$

System (1.88) is also transformed into a similar form,

$$\begin{aligned} &q_t = \nabla \cdot (\nabla q - q\nabla\varphi(v,w)) \\ &v_t = q, \;\; w_t = q \qquad \text{in } \Omega \times (0,T). \end{aligned} \tag{1.91}$$

In 1979, Rascle[265] studied (1.89) for $\varphi(v) = -v$. This system seems to be the case of (1.78) where the diffusion of the second equation is neglected and the chmotaxis term of the right-hand side of the first equation has the negative sign so that the sensitivity $\varphi(v) = -v$ may be regarded as the self-impulsive factor. In the actual interpretation we replace $-v$ by v. Thus u is attarctive to the material v and this v is consumed by u itself. In,[265] global in time solution is obtained using the Lyapunov function in the case of one-space dimension, and a related system of angiogenesis is studied by.[102] This method is applicable to (1.89) in the case that

$$\begin{aligned} &\varphi \in C^3(\mathbf{R}) \\ &q_0, v_0 \in C^{2+\alpha}(\overline{\Omega}) \\ &\frac{\partial q_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \\ &\varphi' \le 0, \quad \varphi'' \ge 0, \end{aligned} \tag{1.92}$$

where $0 < \alpha < 1$. System (1.89) is equivalent to the one studied by,[66]

$$n_t = \nabla \cdot (\nabla n - n\chi(c)\nabla c)$$
$$c_t = -cn \qquad \text{in } \Omega \times (0,T)$$

with

$$\frac{\partial n}{\partial \nu} - \chi(c)\frac{\partial c}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T),$$

where $c, n > 0$, and $\chi = \chi(c)$ is a C^1-function satisfying

$$\chi(c) \geq 0, \qquad c\chi'(c) + \chi(c) \geq 0.$$

In,[66] the global existence in time of the weak solution with the convergence

$$q(\cdot,t) \to \overline{q}_0 \equiv \frac{1}{|\Omega|}\int_\Omega q_0$$

as $t \uparrow +\infty$ is derived formally, see also.[67] Similarly to[265] this property is actually the case in the classical sense when the space-dimension is one, using the continuous embedding, see J.L. Lions,[184]

$$L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^4(0,T;L^\infty(\Omega)).$$

This conclusion is a counter part of the result,[349] that is if $\varphi(v) = v$, we have both global and blowup in finite time solutions depending on their initial data. We note that $\varphi(v) = v$ does not satisfy $\varphi'(v) \leq 0$. This case is a relative to (1.78) with the diffusion term $-\Delta v$ in the second equation neglected.

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$\tau v_t = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).$$

The key structure of (1.89) is

$$\frac{dL}{dt} = -\int_\Omega q^{-1}|\nabla q|^2 + \frac{1}{2}\varphi''(v)q|\nabla q|^2\, dx$$
$$L = \int_\Omega q(\log q - 1) - \frac{1}{2}\varphi'(v)|\nabla v|^2\, dx,$$

and thus L can be a Lyapunov function. We can show the following theorems.[310]

Theorem 1.14. *If (1.92) holds, then there exists a unique solution to (1.89)-(1.90) such that* $q, v \in C^{2+\alpha,1+\alpha/2}(\overline{Q_T})$ *with* $q = q(x,t) > 0$*, provided that* T *is sufficiently small.*

Theorem 1.15. *If (1.92) holds and the space dimension in one, then the solution in the previous theorem exists for any* $T > 0$*. Given* $t_k \uparrow +\infty$ *and* $\delta > 0$*, furthermore, we have* $t_k' \in (t_k - \delta, t_k + \delta)$ *such that*

$$q(\cdot, t_k') \to \overline{q}_0$$

uniformly on $\overline{\Omega}$*.*

Differently from the elliptic-parabolic system,[274] the possibility of the oscillation of $q(\cdot,t)$ as $t \uparrow +\infty$ has not been excluded because of the ODE part of (1.89) even for $n = 1$. Similar results to Theorems 1.14 and 1.15 are valid to (1.91). Actually, we obtain the following theorem.

Theorem 1.16. *If* $0 < n_0, f_0, c_0$ *are* $C^{2+\alpha}$ *on* $\overline{\Omega}$,

$$f_0 > \frac{\beta}{\mu} \qquad \text{on } \overline{\Omega}$$

$$\frac{\partial n_0}{\partial \nu} = \frac{\partial f_0}{\partial \nu} = \frac{\partial c_0}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.93}$$

and the space dimension is one, then there is a unique solution globally in time to (1.88) with the initial boundary condition

$$D\frac{\partial n}{\partial \nu} - \chi(c)n\frac{\partial c}{\partial \nu} - \rho_0 n \frac{\partial f}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

$$n|_{t=0} = n_0, \quad f|_{t=0} = f_0, \quad c|_{t=0} = c_0 \qquad \text{in } \Omega.$$

Any $t_k \uparrow +\infty$ *and* $\delta > 0$, *furthermore, admits* $t_k' \in (t_k - \delta, t_k + \delta)$ *such that*

$$n(\cdot, t_k') \to \overline{n}_0 \equiv \frac{1}{|\Omega|}\int_\Omega n_0$$

uniformly on $\overline{\Omega}$.

In the other case of the first assumption of (1.93),

$$0 < f_0 < \frac{\beta}{\mu}.$$

we obtain *a priori* bounds of the solution for any space dimension under the assumption of

$$(\beta - \mu f_0)^{\gamma/\beta} \ll c_0,$$

and this provides the global in time solution converging to the stationary solution, see.[104]

1.1.11. Geometric Flow

Several geometric flows are known to have the quantized blowup mechanism. In the harmonic heat flow case, the total energy acts as the Lyapunov function. Differently from (1.8), this Lyapunov function is non-negative, and then type (II) blowup rate arises from this non-negativity. In this connection, we note that Herrero-Velázquez' solution to (1.8) has the bounded total free enery, see.[175] Before describing geometric flows, we present an alternative argument to derive (1.61), assuming that the free energy is bounded in (1.8) to illustrate the situation.

For this purpose, we use

$$\frac{d}{dt}\int_\Omega \varphi_{x_0,R}u = -\int_\Omega u\nabla\delta\mathscr{F}(u)\cdot\nabla\varphi_{x_0,R}$$

and derive

$$\left|\frac{d}{dt}\int_\Omega \varphi_{x_0,R}u\right| \leq \left\|u^{1/2}\nabla\delta\mathscr{F}(u)\right\|_2 \cdot \left\|\nabla\varphi_{x_0,R}\right\|_\infty \cdot \|u\|_1$$
$$\leq C\lambda R^{-1}\left\|u^{1/2}\nabla\delta\mathscr{F}(u)\right\|_2.$$

If

$$\lim_{t\uparrow T}\mathscr{F}(u(\cdot,t)) > -\infty, \tag{1.94}$$

is the case, then it holds that

$$\int_0^T \left\|u^{1/2}\nabla\delta\mathscr{F}(u)(\cdot,t)\right\|_2^2 dt < +\infty$$

by the second equality of (1.20), and, therefore, we can replace (1.39) by

$$\begin{aligned}&\left|\left\langle\varphi_{x_0,R},\mu(dx,t)\right\rangle - \left\langle\varphi_{x_0,R},\mu(dx,T)\right\rangle\right| \\ &\quad\leq C\lambda R^{-1}\int_t^T \left\|u^{1/2}\nabla\delta\mathscr{F}(u)(\cdot,s)\right\|_2 ds \\ &\quad\leq C\lambda R^{-1}(T-t)^{1/2}\left\{\int_t^T \left\|u^{1/2}\nabla\delta\mathscr{F}(u)(\cdot,s)\right\|_2^2 ds\right\}^{1/2}\end{aligned}$$

for $x_0 \in \mathscr{S}$. This inequality guarantees

$$\left\langle\varphi_{x_0,bR(t)},\mu(dx,t)\right\rangle = m_*(x_0) + o(1)$$

as $t \uparrow T$ for $R(t) = (T-t)^{1/2}$, where $b > 0$ is arbitrary. Then, $\zeta(dy,s)$ generated by (1.41) from $s_k \uparrow +\infty$ has the form

$$\zeta(dy,s) = m_*(x_0)\delta_0(dy)$$

which implies that (1.61), and in particular, any $x_0 \in \mathscr{S}$ is of type (II). □

Harmonic Heat Flow

As is described above, formation of the quantized collapse is observed also in the harmonic heat flow.[296] If we take the flat torus $\Omega = \mathbf{R}^2/a\mathbf{Z}\times b\mathbf{Z}$, $a,b>0$ and the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ as the domain and the target, respectively, then this flow is described by

$$u = u(x,t) : \Omega \times [0,T) \to S^{n-1} \subset \mathbf{R}^n$$

satisfying

$$\begin{aligned}&u_t - \Delta u = u|\nabla u|^2 \\ &|u| = 1 \qquad\qquad \text{in } \Omega\times(0,T).\end{aligned} \tag{1.95}$$

In this case, it holds that

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = \int_\Omega u\cdot u_t|\nabla u|^2$$
$$= \frac{1}{2}\int_\Omega \left(\frac{\partial}{\partial t}|u|^2\right)|\nabla u|^2 = 0$$

and hence

$$\frac{dE}{dt} = -\|u_t\|_2^2 \le 0$$
$$E = \frac{1}{2}\|\nabla u\|_2^2. \tag{1.96}$$

Thus, the the above E casts the Lyapunov function.
The stationary solution to (1.95) is called the *harmonic map*. In the general setting, we take the m-dimensional compact Riemannian manifold (Ω, g) and the compact Riemannian manifold N without boundaries. By Nash's theorem this N is isometrically imbedded in $\mathbf{R}^n$ for large n. We define the Sobolev space composed of a class of the mappings from Ω to N provided with the finite energy as in the previous section, that is

$$H^1(\Omega,N) = \left\{u\in H^1(\Omega,\mathbf{R}^n) \mid u\in N,\ \text{ a.e. on } \Omega\right\}$$
$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dv_\Omega,$$

where dv_Ω is a volume element of (Ω,g). We call a map $u\in H^1(\Omega,N)$ (weakly) harmonic if

$$\left.\frac{d}{d\varepsilon}E(\Pi(u+\varepsilon\phi))\right|_{\varepsilon=0} = 0$$

for any $\phi\in C_0^\infty(\Omega,\mathbf{R}^n)$, where $\Pi : U\to N$ is a smooth nearest point projection from some tubular neighborhood U of N to N. This relation is equivalent to saying that u is a weak solution of the Euler-Lagrange equation

$$-\Delta u = A(u)(\nabla u,\nabla u) \qquad \text{on } \Omega, \tag{1.97}$$

sometimes called the *harmonic map equation*, where Δ and $A(u)(\cdot,\cdot)$ denote the Laplace-Beltrami operator on (M,g) and the second fundamental form of the imbedding $N\hookrightarrow\mathbf{R}^n$ at $y\in N$, respectively. Harmonic heat flow is the mapping $u = u(x,t) : \Omega\times[0,T)\to N\subset\mathbf{R}^n$ satisfying

$$u_t = -\delta E(u)$$

and, therefore, the harmonic map is regarded as a stationary state of the harmonic heat flow. First, we have the energy quantization for the sequence of harmonic maps.

Theorem 1.17 ([79,158,207,260–262,296,330]). *Let $\{u_k\}_k$ be a harmonic map sequence satisfying $\sup_k E(u_k) < +\infty$. Then, passing to a subsequence we assume $u_k \rightharpoonup u$ in $H^1(\Omega,N)$ weakly to some map $u\in H^1(\Omega,N)$. This u is a harmonic map, and there exist*

- *p-sequences of points $\{x_k^1\},\cdots,\{x_k^p\}$ in Ω*

- *p-sequences of positive numbers* $\{\delta_k^1\}, \cdots, \{\delta_k^p\}$ *converging to* 0
- *p-non-constant harmonic maps* $\{\omega^1\}, \cdots, \{\omega^p\} : S^2 \to N$

satisfying the following:

$$\lim_{k\to\infty} E(u_k) = E(u) + \sum_{j=1}^{p} E_0(\omega^j)$$
$$\lim_{k\to\infty} \max_{i\neq j} \left\{ \frac{\delta_k^i}{\delta_k^j}, \frac{\delta_k^j}{\delta_k^i}, \frac{\left|x_k^i - x_k^j\right|}{\delta_k^i + \delta_k^j} \right\} = +\infty$$
$$\lim_{k\to\infty} \left\| u_k - u - \sum_{j=1}^{p} \left\{ \omega^j \left(\frac{\cdot - x_k^j}{\delta_k^j} \right) - \omega^j(\infty) \right\} \right\|_{H^1(\Omega,N)} = 0, \tag{1.98}$$

where

$$E_0(\omega) = \frac{1}{2} \int_{S^2} |\nabla \omega|^2 \, dv_{S^2}.$$

The first equality of (1.98) is an energy identity which says that there is no unaccounted energy loss during the iterated rescaling process near the point of singularity, sometimes referred to as the bubbling process, and that the only reason for failure of strong convergence to the weak limit is the formation of several bubbles due to the non-constant harmonic maps $\omega^j : S^2 \to N$, $j = 1, \ldots, p$. Differently from (1.25), there is a possibility that some of $\left\{x_k^j\right\}_k$, $j = 1, \ldots, p$, converge to the same point, and this process is classified into two cases - the separated bubbles and the bubbles on bubbles.[308]

The ε-regularity and the monotonicity formula are known to the harmonic heat flow, similarly to the simplified system of chemotaxis (1.8). In the case of (1.95) for $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$, first, there is $\varepsilon_0 > 0$ such that $u = u(x,t)$ is smooth in $B_{R/2} \times [0,T]$, provided that

$$\sup_{t\in[0,T]} E(u(\cdot,t), B_R) < \varepsilon_0,$$

where

$$B_R = B(0,R)$$
$$E(u,R) = \frac{1}{2} \int_{B_R} |\nabla u|^2$$
$$E_0 = \frac{1}{2} \|\nabla u_0\|_2^2.$$

Second,

$$E(u(\cdot,T), B_R) \leq E(u_0, B_{2R}) + CE_0T/R^2$$

holds with $C > 0$. These analytic structures guarantee that there is a weak solution global in time with a finite number of sigular points in $\Omega \times [0,+\infty)$, and then we will obtain a homotopy between the initial state and the expected ultimate state, see.[295,296]

Similarly to the chemotaxis system (1.8) with (1.94), the non-negativity of E implies

$$\frac{1}{2}\sup_{t\in[0,T)}\|\nabla u(\cdot,t)\|_2^2+\int_0^T\|u_t(\cdot,s)\|_2^2\,ds \le \frac{1}{2}\|\nabla u_0\|_2^2 = E_0 \tag{1.99}$$

by (1.96). Then, there is $t_k \uparrow T$ satisfying

$$(T-t_k)\,\|u_t(t_k)\|_2^2 \to 0 \tag{1.100}$$

because otherwise it holds that

$$\int_0^T \|u_t(\cdot,s)\|_2^2 ds = +\infty,$$

a contradiction. For this $u_k = u(\cdot,t_k)$, it is known, see,[324] that the conclusion of Theorem 1.17 arises. The hyper-parabola also works, and thus there are similarities and differences between (1.8) and (1.95).

More precisely, we have

$$\begin{aligned}\int_\Omega |u_t|^2\varphi^2+\int_\Omega \nabla u\cdot\nabla(u_t\varphi^2) &= \int_\Omega u_t\cdot u|\nabla u|^2\varphi^2\\ &= \frac{1}{2}\int_\Omega\left[\frac{\partial}{\partial t}|u|^2\right]|\nabla u|^2\varphi^2=0\end{aligned}$$

with

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla(u_t\varphi^2) &= \int_\Omega(\nabla u\cdot\nabla u_t)\varphi^2+\int_\Omega[(u_t\cdot\nabla)u]\cdot\nabla\varphi^2\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u|^2\varphi^2+\int_\Omega[(u_t\cdot\nabla)u]\cdot\nabla\varphi^2\end{aligned}$$

from (1.95), and, therefore,

$$\int_\Omega |u_t|^2\varphi^2+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u|^2\varphi^2+\int_\Omega[(u_t\cdot\nabla)u]\cdot\nabla\varphi^2=0$$

for each $\varphi\in C^1(\Omega)$ which implies that

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u|^2\varphi^2\right| &\le \|u_t\|_2^2\|\varphi\|_\infty^2+\|u_t\|_2\cdot\left\{\int_\Omega|\nabla u|^2|\nabla\varphi^2|^2\right\}^{1/2}\\ &\le \|u_t\|_2^2\|\varphi\|_\infty^2+\sqrt{2}\|u_t\|_2\cdot E_0^{1/2}\|\nabla\varphi^2\|_\infty,\end{aligned} \tag{1.101}$$

and then it follows that

$$\int_0^T\left|\frac{d}{dt}\int_\Omega|\nabla u|^2\varphi^2\right|dt<+\infty.$$

Thus

$$\mu(dx,t)=|\nabla u(x,t)|^2dx\in C_*([0,T],\mathscr{M}(\Omega))$$

follows againg from (1.99). Using $t_k\uparrow T$ satisfying (1.100), we obtain

$$\mu(dx,T)=\sum_{x_0\in\mathscr{S}} m(x_0)\delta_{x_0}(dx)+f(x)dx \tag{1.102}$$

with a finite set $\mathscr{S}$ and $0 \leq f = f(x) \in L^1(\Omega)$. The above described result[324] guarantees the "energy quantization." More precisely, $m(x_0) > 0$ is a finite sum of the energies of non-constant harmonic maps: $S^2 \to N$.
Inequality (1.101) implies also

$$\left| \int_\Omega |\nabla u(\cdot,t)|^2 \varphi_{x_0,R}^2 - \int_\Omega f(x) \varphi_{x_0,R}^2 - m(x_0) \right| \leq 2C \int_t^T \|u_t(\cdot,s)\|_2^2 ds$$
$$+2\sqrt{2} E_0^{1/2} \left\{ \int_t^T \|u_t(\cdot,s)\|_2^2 ds \right\}^{1/2} \cdot C(T-t)^{1/2}/R$$

for $x_0 \in \mathscr{S}$ with $C > 0$ independent of $0 < R \ll 1$. Then it follows that

$$\lim_{t \uparrow T} \int_\Omega |\nabla u(\cdot,t)|^2 \varphi_{x_0,bR(t)}^2 = m(x_0)$$

from (1.99) again, where $b > 0$ is arbitrary and $R(t) = (T-t)^{1/2}$. The hyper-parabola thus arises and we obtain type (II) blowup rate at each $x_0 \in \mathscr{S}$ similarly to (1.8) provided with (1.94).

Normalized Ricci Flow

The normalized Ricci flow describes an evolution in time of the metric $g = g(t)$ on a compact Riemannian manifold. If Ω is a compact Riemannian surface, this flow is given by

$$\frac{\partial g}{\partial t} = (r - R) g, \qquad t > 0 \tag{1.103}$$

where $R = R(\cdot,t)$ stands for the scalar curvature of $(\Omega, g(t))$ and $r = r(t)$ represents the average scalar curvature given by

$$r = \frac{\int_\Omega R(\cdot,t)\, d\mu_t}{\int_\Omega d\mu_t}$$

with the volume element $\mu = \mu_t$. Hamilton[131] introduced the above flow to approach the Poincaré conjecture, and this idea has the right.[253–255] In the case that Ω is a comapct Riemannian surface is described in,[132] and it is shown that the solution to (1.103) exists globally in time, converges in C^∞-topology as $t \uparrow +\infty$, and the scalar curvature of the limit metric is constant. Here, we describe an arguement using the analytic form of (1.103).
First, we obtain

$$\frac{\partial R}{\partial t} = \Delta_t R + R(R - r)$$

and, therefore, $R(\cdot,t) > 0$ everywhere on Ω follows from $R(\cdot,0) > 0$ everywhere on Ω, where Δ_t denotes the Laplace-Beltrami operator associated with $g = g_t$. Henceforth, we deal with this case. Then, from Gauss-Bonnet's theorem there follows

$$\int_\Omega R(\cdot,t)\, d\mu_t = 4\pi\chi(\Omega). \tag{1.104}$$

Here, $\chi(\Omega) = 2 - 2k(\Omega)$ stands for the Euler characteristic of Ω, and hence $k(\Omega)$ is the genus of Ω. Since $R(\cdot,t) > 0$ in Ω, this formula gives $k(\Omega) = 0$, and then the uniformization theorem reduces the problem to the case that

$$\Omega = S^2$$
$$g(t) = e^{w(\cdot,t)} g_0,$$

where S^2 is the two dimensional sphere, g_0 is its standard metric, and $w = w(\cdot,t)$ is a smooth function.

In this case, the scalar curvature R_0 corresponding to the metric g_0 is a constant, and it is related to $R = R(\cdot,t)$ through

$$R = e^{-w}(-\Delta w + R_0) \tag{1.105}$$

with $\Delta = \Delta_{g_0}$. Here, we obtain

$$\int_{S^2} R(\cdot,t)\, d\mu_t = 8\pi \tag{1.106}$$

by (1.104) and hence

$$r = \frac{8\pi}{\int_{S^2} d\mu_t} = \frac{8\pi}{\int_{S^2} e^w dx}, \tag{1.107}$$

setting $dx = d\mu_{g_0}$. Finally, we have

$$|S^2| R_0 = 8\pi \tag{1.108}$$

because (1.106) holds and R_0 is a constant.

By pluging (1.105) into (1.103) and using (1.107), (1.108), we end up with

$$\frac{\partial e^w}{\partial t} = \Delta w + 8\pi \left(\frac{e^w}{\int_{S^2} e^w\, dx} - \frac{1}{|S^2|} \right) \qquad \text{in } S^2 \times (0,T). \tag{1.109}$$

The result in[132] thus reads as follows. The solution to (1.109) exists globally in time and $w(\cdot,t) \to w_\infty$ as $t \uparrow +\infty$ in C^∞-topology, with w_∞ standing for a stationary solution:

$$-\Delta w_\infty = 8\pi \left(\frac{e^{w_\infty}}{\int_{S^2} e^{w_\infty}\, dx} - \frac{1}{|S^2|} \right) \qquad \text{in } S^2. \tag{1.110}$$

From the viewpoint of dynamical systems, the proof of[132] consists of three ingredients: extension of the solution globally in time, compactness of the orbit, and uniqueness of the ω-limit set. All the steps are based on the geometric structure of (1.103), involving Harnack's inequality for the scalar curvature, monotonicity of an awkward geometric quantity called "entropy", soliton solutions of the Ricci flow, the modified Ricci flow, and so forth. There are, however, several complementary analytic arguments.

First, the third step originally achieved by modifying (1.103) via a transformation group may be replaced by the uniqueness of the solution to

$$-\Delta w = 8\pi \left(\frac{e^w}{\int_{S^2} e^w\, dx} - \frac{1}{|S^2|} \right) \quad \text{in } S^2$$
$$\int_{S^2} w = 0. \tag{1.111}$$

In fact, since

$$\frac{d}{dt}\int_{S^2} e^w\,dx = 0$$

follows from (1.109), the stationary solution $w = w_\infty$ to (1.109) constitutes of (1.110) with

$$\int_{S^2} e^{w_\infty}\,dx = \int_{S^2} e^{w_0}\,dx.$$

This w_∞, on the other hand, must be a constant, provided that only the trivial solution $w = 0$ is admitted to (1.111). This property is actually the case because the metric on S^2 with constant Gaussian curvature is the standard one. Thus the steady state of (1.109) is unique:

$$w_\infty = \frac{1}{|\Omega|}\int_{S^2} e^{w_0} dx.$$

Next, a gradient estimate of the form

$$|\nabla_{S^2} w| \le C \tag{1.112}$$

is obtained using the symmetry of S^2, that is the moving sphere method, with C depending only on $w_0 = w(\cdot, 0)$, see.[18] This estimate, combined with the argument in[350] based on Harnack's inequality, induces also global in time existence of the solution to (1.109) and also the compactness of the orbit.
Then, what happens to

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) \qquad \text{in } \Omega \times (0,T), \tag{1.113}$$

where $\lambda > 0$ is a constant and Ω is a compact Riemannian surface without boundary? Unless $\lambda = 8\pi$ and $\Omega = S^2$, it is not a normalized Ricci flow (1.109) any more. An estimate like (1.112) may not be obtained because of the lack of the symmetry of Ω. The arguments of[132] using the geometric structure such as the covariant and Lie derivatives, Bochner-Weitzenböck's formula, and so forth, see,[63] are also invalid, and even global in time existence of the solution is not obvious.
Similarly to (1.106), however, we have

$$\frac{d}{dt}\int_\Omega e^w = 0,$$

and, therefore,

$$r = \frac{\lambda}{\int_\Omega e^w}$$

is a constant. Under the change of variables $u = re^w$ and $t = r^{-1}\tau$, and writing t for τ, problem (1.113) is transformed into

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \tag{1.114}$$

with

$$\int_\Omega u(\cdot,t) = \lambda. \tag{1.115}$$

This form, (1.114), is a non-local perturbation of the logarithmic diffusion equation

$$u_t = \Delta \log u \qquad \text{in } \Omega \times (0,T) \tag{1.116}$$

which also describes the evolution of surfaces by Ricci flow.[132] There are also some other physical problems which could be described by (1.116); the spread over a thin colloidal film at a flat surface,[36,74,336] the modelling of the expansion of a thermalized cloud of electrons,[189] and the central limit approximation to the Calerman's model of the Boltzmann equation.[177,188] Due to a variety of applications logarithmic diffusion equation (1.116) has attracted the interest of many researchers, see for instance.[326]

The spatially homogeneous part of (1.114) is formulated by the non-local ODE,

$$u_t = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)$$

but the non-local term is removable by (1.115). The regions

$$\Omega_+(t) = \{x \in \Omega \mid u(x,t) > \lambda/|\Omega|\}$$
$$\Omega_-(t) = \{x \in \Omega \mid u(x,t) < \lambda/|\Omega|\}$$

are invariant in t, and, therefore, the *extinction* $\lim_{t\uparrow T} \inf_\Omega u(\cdot,t) = 0$ arises before the blowup $\lim_{t\uparrow T} \sup_\Omega u(\cdot,t) = +\infty$ comes. This fact is also known to (1.116), and we obtain

$$\|u(\cdot,t)\|_1 = \|u_0\|_1 - 4\pi t$$

in the case of $\Omega = \mathbf{R}^2$, see[327] and the references therein.

Equation (1.114), however, may be written as

$$\begin{aligned} & u_t = \Delta(\log u - v) \\ & -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad && \text{in } \Omega \times (0,T) \\ & \int_\Omega v = 0, && 0 < t < T, \end{aligned} \tag{1.117}$$

which is to be compared with the Smoluchowski-Poisson equation (1.8) formulated by (1.19),

$$\begin{aligned} & u_t = \nabla \cdot (u\nabla(\log u - v)) \\ & -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad && \text{in } \Omega \times (0,T) \\ & \int_\Omega v = 0 && 0 < t < T. \end{aligned} \tag{1.118}$$

Actually, (1.114) is again a model (B) equation formulated by Helmholtz' free energy

$$\mathscr{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle v, u\rangle$$

and from this fact it is easy to derive the total mass preserving and the decrease of the free energy

$$\frac{d}{dt}\int_\Omega u = 0$$
$$\frac{d}{dt}\mathscr{F}(u) = -\int_\Omega |\nabla(\log u - v)|^2 \le 0.$$

Concerning the Smoluchowski-Poisson equation (1.118), there is, however, a fundamental property obtained by the method of symmetrization that is (1.35) which results in the formation of collapse with the quantized mass. This control in time of the local L^1-norm of the solution $u = u(\cdot,t)$ to (1.117) is not available. The equivalent form (1.114), however, is provided with the comparison principle thanks to (1.115). An important consequence of this property is the monotonicity formula of Benilan's type,

$$\frac{\partial}{\partial t}\left(\frac{u}{e^t - 1}\right) \le 0, \tag{1.119}$$

which guarantees the point-wise convergence of

$$u(x,T) = \lim_{t\uparrow T} u(x,t)$$

in the case of $T = T_{\max} < +\infty$, where $T_{\max} \in (0,+\infty]$ denotes the existence time of the solution. The following theorems are obtained by this structure and Fontana's Trudinger-Moser inequality,[101]

$$\inf\left\{ J_{8\pi}(v) \mid v \in H^1(\Omega),\ \int_\Omega v = 0 \right\} > -\infty$$
$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1),$$

see.[164]

Theorem 1.18. *If $0 < \lambda < 8\pi$, then the solution $u = u(\cdot,t)$ to (1.114) satisfies the uniform estimates*

$$0 < u(x,t),\ u(x,t)^{-1} \le C \qquad \text{in } \Omega\times(0,T).$$

The solution $w = w(\cdot,t)$ to (1.113), therefore, exists globally in time, and the orbit $\mathscr{O} = \{w(\cdot,t)\}_{t\ge 0}$ is compact in $C(\Omega)$. The ω-limit set of $\mathscr{O}$ is thus non-empty, connected, comapct, and contained in the set of statoinary soltions, and in particular, any $t_k \uparrow +\infty$ admits $\{t_k'\} \subset \{t_k\}$ and $w_\infty = w_\infty(x)$ satsifying

$$-\Delta w = \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) \qquad \text{in } \Omega$$
$$\int_\Omega e^w = \int_\Omega e^{w_0} \tag{1.120}$$

such that $w(\cdot,t_k') \to w_\infty$ uniformly on Ω.

Theorem 1.19. *Even for $\lambda = 8\pi$, the solution $u = u(\cdot,t)$ to (1.114) exists globally in time. There is $t_k \uparrow \infty$ and w_∞ satisfying (1.120) such that $w(\cdot,t_k) \to w_\infty$ in $C(\Omega)$.*

The asymptotic behavior $w = w(\cdot,t)$ to $\lambda \le 8\pi$ is thus controlled by the uniqueness and the non-degeneracy of the stationary solution. For this topic, we have[50,60,182] and[183] when $\Omega = S^2$ and $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$, respectively.

1.1.12. Summary

We have studied the chemotaxis system and observed that the formation of self-assembly is realized in thermally and materially closed system, whereby the principle of nonlinear spectral mechanics is efficient.

(1) The simplified system of a chemotax is the model (B) equation derived from Helmholtz' free energy, and, conseqeuntly, the stationary state is described by the Euler-Lagrange equation derived from this free energy.
(2) Using the field component, this stationary state is re-formulated as a nonlinear eigenvalue problem with non-local term.
(3) Then, the total set of stationary solutions controls the global dynamics.
(4) It is described by the quantized blowup mechanism, which results in the free energy transmission.
(5) The key profile of the solution to the Smoluchowski-Poisson equation is obtained by a weak formulation and by using a symmetry of the Green's funtion, that is bounded in time variation of the local L^1-norm. Then, we apply a hierarchical argument to prove mass quantization.
(6) Other non-stationary problems sharing the same stationary problem arise in biology and geometry which, however, obey different features of dis-quantized blowup mechanism.

1.2. Toland Duality

The collapse mass quantization, (1.13) to (1.8), has its origin in a stationary state (1.25) which provides a variational structure of duality between the particle density u and the field distribution v. These variational problems each split into components and at which point, then, the linearized stability of the stationary state implies a dynamical stability. In this section we formulate the above duality observed in the full system of chemotaxis; we thus develop the abstract theory and then examine the continuity of the entropy functional.

1.2.1. Exponential Nonlinearity

Besides a stationary state of chemotaxis, the additional problem of an elliptic eigenvalue with a non-local term becomes evident whereby the two-dimensional Laplacian competes with an exponential nonlinearity as shown in Witten-Taube's gauge of Shrödinger equations,[317,318,337] and Onsager's theory of mean field stationary turbulence.[241]

This eigenvalue problem is involved in a complex analysis and a theory of surfaces,[303] and its quantized blowup mechanism was first observed to

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \qquad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{1.121}$$

see.[213] More precisely, in this problem, non-compact solution sequence arises only at the quantized value of $\lambda \in 8\pi\mathbf{N}$, and the location of the blowup points is controlled by the Green's function. This quantized blowup mechanism admits several approaches to the solution set to (1.121), the singular perturbation,[15] the topological degree calculation,[54,55,180] and so forth. The uniqueness of the solution, on the other hand, holds if $0 < \lambda < 8\pi$, see.[47,301,308] It is proven by the bifurcation analysis combined with the isoperimetric inequality on surfaces,[14] see §2.3.2 for details.

Problem (1.121) is the Euler-Lagrange equation to the variational functional

$$\mathscr{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1) \tag{1.122}$$

defined for $v \in H_0^1(\Omega)$, and the Morse index of its critical point v indicates the maximum dimension of the linear subspace where the associated quadratic form

$$Q(w,w) = \frac{1}{2}\frac{d^2}{ds^2}\mathscr{J}_\lambda(v+sw)\bigg|_{s=0}$$

defined for $w \in H_0^1(\Omega)$ is negative definite.

In,[301] the Morse index of the solution $v = \overline{v}$ to (1.121) is shown to be equal to the number of eigenvalues in $\mu < 1$ minus one of the eigenvalue problem

$$\begin{aligned} -\Delta\phi &= \mu\overline{u}\phi && \text{in } \Omega \\ \phi &= \text{constant} && \text{on } \partial\Omega \\ \int_{\partial\Omega}\frac{\partial\phi}{\partial\nu} &= 0, \end{aligned} \tag{1.123}$$

where

$$\overline{u} = \frac{\lambda e^{\overline{v}}}{\int_\Omega e^{\overline{v}}}. \tag{1.124}$$

Here, we emphasize that $\mu = 0$ is the first eigenvalue of (1.123) with the corresponding eigenfuction being a constant.

Problem (1.121) is also the stationary state of a similar system to (1.8),

$$\begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v) \\ 0 &= \Delta v + u && \text{in } \Omega\times(0,T) \\ \frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu} &= v = 0 && \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{1.125}$$

subject to the decrease of the free energy defined by

$$\mathscr{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\left\langle(-\Delta_D)^{-1}u, u\right\rangle. \tag{1.126}$$

Here and henceforth, we write $v = (-\Delta_D)^{-1}u$ if and only if

$$-\Delta v = u \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

This stationary state is nothing but the Euler-Lagrange equation of the variational problem $\delta\mathscr{F}(u) = 0$ constrained by $\|u\|_1 = \lambda$, that is,

$$(-\Delta_D)^{-1}u = \log u + \text{constant} \quad \text{in} \quad \Omega, \qquad \|u\|_1 = \lambda \tag{1.127}$$

for $u = u(x) > 0$. From this variational structure, it is natural to define the Morse index of the solution u to (1.127) by the dimension of the maximal linear subspace where the associated quadratic form

$$R(\varphi,\varphi) = \frac{1}{2}\,\frac{d^2}{ds^2}\mathscr{F}(u+s\varphi)\bigg|_{s=0}, \qquad \varphi \in L^1(\Omega),\ \int_\Omega \varphi = 0,$$

is negative.
This index again, is equal to the number of eigenvalues in $\mu < 1$ minus 1 of (1.123), see,[338] and thus, the problems (1.121) and (1.127) on v and u are not only equivalent through (1.124) and

$$v = (-\Delta_D)^{-1}u, \tag{1.128}$$

but also their Morse indices coincide each other. This spectral equivalence between $\mathscr{J}_\lambda(v)$ and $\mathscr{F}(u)$, $u \geq 0$, $\|u\|_1 = \lambda$ is the starting point of Chapter 9 of.[304] It is actually used to show the dynamical stability of the linearly stable stationary solution,[302] see §§1.2.8 and 2.3.4.

1.2.2. Full System of Chemotaxis

The above spectral equivalence of variations, however, is a consequence of the general theory of *dual variation*. This structure has been suggested in §1.1.10 for the full system of chemotaxis (1.78), namely, the Lyapunov function $L(u,v)$ defined by (1.79). We are now describing the theory, taking the system

$$\begin{aligned} &u_t = \nabla\cdot(\nabla u - u\nabla v) \\ &\tau v_t = \Delta v + u && \text{in } \Omega\times(0,T) \\ &\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = v = 0 && \text{on } \partial\Omega\times(0,T) \end{aligned} \tag{1.129}$$

for simplicity.
First, system (1.129) is also provided with the Lyapunov function, $L = L(u,v)$ defined by

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v,u\rangle. \tag{1.130}$$

Using this Lyapunov function, (1.129) is written as

$$\begin{aligned} &u_t = \nabla\cdot(u\nabla L_u(u,v)) \\ &\tau v_t = -L_v(u,v) && \text{in } \Omega\times(0,T) \\ &u\frac{\partial}{\partial \nu}L_u(u,v) = v = 0 && \text{on } \partial\Omega\times(0,T), \end{aligned}$$

and it holds that

$$\frac{d}{dt}\int_\Omega u = 0$$
$$\frac{d}{dt}L(u,v) + \tau \|v_t\|_2^2 + \int_\Omega u\left|\nabla(\log u - v)\right|^2 = 0, \tag{1.131}$$

where $(u,v) = (u(\cdot,t), v(\cdot,t))$ is the solution to (1.129) and $\langle\ ,\ \rangle$ denotes the duality,

$$\langle v,u\rangle = \int_\Omega uv.$$

Equality (1.128) holds in the simplified system (1.125), and in this case the above $L(u,v)$ is reduced to the free energy defined by (1.126),

$$L|_{v=(-\Delta_D)^{-1}u} = \mathscr{F}. \tag{1.132}$$

We have, on the other hand,

$$u = \frac{\lambda e^v}{\int_\Omega e^v} \tag{1.133}$$

in the stationary state of (1.129) because

$$\log u - v = \text{constant}$$

follows from (1.131) with

$$\frac{d}{dt}L(u,v) = 0,$$

where $\|u\|_1 = \lambda$. Then, (1.121) is obtained by putting (1.133) and $v_t = 0$ in the second equation of (1.129). The associated variational function $\mathscr{J}_\lambda$ defined by (1.122) is obtained similarly to (1.132):

$$L|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} = \mathscr{J}_\lambda. \tag{1.134}$$

The "semi-stationary" state of (1.129) with (1.133) is nothing but the non-local parabolic equation (1.81). Henceforth, we call (1.132) and (1.134) the *unfolding Legendre transformation*.

We have, on the other hand, the *minimality* indicated by

$$L(u,v) \geq \max\left\{\mathscr{F}(u), \mathscr{J}_\lambda(v)\right\}, \tag{1.135}$$

where $\|u\|_1 = \lambda$. In fact, the first inequality is a direct consequence of Schwarz' inequality, while the second inequality is proven by Jensen's inequality. It is applicable to infer the global existence of the solution to (1.125) or (1.129) in the case of $\lambda = \|u_0\|_1 < 8\pi$, but is regarded as a dual form of the Trudinger-Moser inequality, see[304] and also §1.2.8.

We remind the reader that these structures are valid also to (1.78). We take $L = L(u,v)$ of (1.130) for $v \in H^1(\Omega)$ with $\int_\Omega v = 0$. Then, it holds that

$$L|_{v=(-\Delta_{JL})^{-1}u} = \mathscr{F}(u)$$
$$\equiv \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta_{JL})^{-1}u, u\rangle,$$

$$
\begin{aligned}
L|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} &= \mathscr{J}_\lambda(v) \\
&\equiv \frac{1}{2}\|\nabla v\|_2^2 - \lambda\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1),
\end{aligned}
$$

and (1.135). Here and henceforth, $v = (-\Delta_{JL})^{-1}u$ if and only if

$$
v = (-\Delta_N)^{-1}(u - \overline{u}), \qquad \overline{v} = 0,
$$

that is

$$
\begin{aligned}
&-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \\
&\int_\Omega v = 0.
\end{aligned}
$$

More precisely, *dual variation* guarantees the splitting of the stationary state of each component provided with the variational and dynamical equivalence and then, the above mentioned unfolding-minimality arises in the context of convex analysis.

1.2.3. Lagrangian

We are in position to formulate the general theory of a dual variation. Let X be a Banach space over $\mathbf{R}$. Its dual space and the duality paring are denoted by X^* and $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{X,X^*}$, respectively.

Given $F : X \to [-\infty, +\infty]$, we define its Legendre transformation $F^* : X^* \to [-\infty, +\infty]$ by

$$
F^*(p) = \sup_{x\in X}\left\{\langle x, p\rangle - F(x)\right\}, \qquad p \in X^*.
$$

Then, Fenchel-Moreau's theorem guarantees that if

$$
F : X \to (-\infty, +\infty]
$$

is proper, convex, lower semi-continuous, then so is

$$
F^* : X^* \to (-\infty, +\infty],
$$

and the second Legendre transformation defined by

$$
F^{**}(x) = \sup_{p\in X^*}\left\{\langle x, p\rangle - F^*(p)\right\}, \qquad x \in X
$$

is equal to $F(x)$, see.[30,83] Here and henceforth, we say that $F : X \to (-\infty, +\infty]$ is *proper* if its effective domain defined by

$$
D(F) = \{x \in X \mid F(x) \in \mathbf{R}\}
$$

is not empty; *convex* if

$$
F(\theta x + (1-\theta)y) \le \theta F(x) + (1-\theta)F(y)
$$

for any $x, y \in X$ and $0 < \theta < 1$; and *lower semi-continuous* if

$$
F(x) \le \liminf_k F(x_k),
$$

provided that $x_k \to x$ in X.
Let $F, G : X \to (-\infty, +\infty]$ be proper, convex, lower semi-continuous, with the effective domains $D(F)$ and $D(G)$, respectively.

$$D(F) = \{x \in X \mid F(x) < +\infty\}$$
$$D(G) = \{x \in X \mid G(x) < +\infty\},$$

and let

$$\varphi(x,y) = F(x+y) - G(x) \tag{1.136}$$

for $x \in D(G)$. Then,

$$y \in X \mapsto \varphi(x,y) \in (-\infty, +\infty]$$

is proper, convex, lower semi-continuous, and its Legendre transformation is defined by

$$L(x,p) = \sup_{y \in X} \{\langle y, p\rangle - \varphi(x,y)\} \tag{1.137}$$

for $p \in X^*$. Thus

$$L(x,\cdot) : X^* \to (-\infty, +\infty]$$

is proper, convex, lower semi-continuous. Sometimes $-L(x,p)$ is referred to as the *Lagrangian*, but in this book, $L(x,p)$ is called the Lagrangian directly.
If $(x,p) \in D(G) \times X^*$, then it holds that

$$\begin{aligned} L(x,p) &= \sup_{y \in X} \{\langle y+x, p\rangle - F(x+y) + G(x) - \langle x, p\rangle\} \\ &= F^*(p) + G(x) - \langle x, p\rangle . \end{aligned} \tag{1.138}$$

Putting $L(x,p) = +\infty$ for $x \notin D(G)$, on the other hand, we obtain (1.138) for any $(x,p) \in X \times X^*$. Thus we define

$$J^*(p) = \begin{cases} F^*(p) - G^*(p), & p \in D(F^*) \\ +\infty, & \text{otherwise} \end{cases} \tag{1.139}$$

for $p \in X^*$. Then, it holds that

$$\begin{aligned} \inf_{x \in X} L(x,p) &= F^*(p) - \sup_{x \in X} \{\langle x, p\rangle - G(x)\} \\ &= F^*(p) - G^*(p) = J^*(p) \end{aligned}$$

for $p \in D(F^*)$. Let us note that this relation is valid even to $p \notin D(F^*)$ by (1.138) and (1.139).
Similarly, we define

$$J(x) = \begin{cases} G(x) - F(x), & x \in D(G) \\ +\infty, & \text{otherwise} \end{cases} \tag{1.140}$$

for $x \in X$, and obtain

$$\begin{aligned} \inf_{p \in X^*} L(x,p) &= G(x) - \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\} \\ &= G(x) - F^{**}(x) = J(x) \end{aligned}$$

for $x \in D(G)$, which is valid even to $x \notin D(G)$ by (1.138) and (1.140). Thus, we have

$$D(J) = \{x \in X \mid J(x) \neq \pm\infty\} = D(G) \cap D(F)$$
$$D(J^*) = \{p \in X^* \mid J^*(p) \neq \pm\infty\} = D(G^*) \cap D(F^*)$$

and

$$\begin{aligned} &\inf_{x \in X} L(x,p) = J^*(p),\ p \in X^* \\ &\inf_{p \in X^*} L(x,p) = J(x),\ x \in X. \end{aligned} \tag{1.141}$$

Relation (1.141) implies

$$\inf_{(x,p) \in X \times X^*} L(x,p) = \inf_{p \in X^*} J^*(p) = \inf_{x \in X} J(x), \tag{1.142}$$

called the *Toland duality*.[322,323] Here, we call $\langle v, u \rangle$ of (1.138) the *hook term*. The functional

$$\varphi(x,y) = F(x+y) - G(x) : X \times X \to [-\infty, +\infty]$$

of (1.136) is called the *cost function*, which is convex in y-component.
Differently from the above properties of the Toland duality, the cost function

$$\varphi = \varphi(x,y) : X \times X \to [-\infty, +\infty]$$

of the *Kuhn-Tucker duality* has the property that $x \in X \mapsto \varphi(x,y)$ and $y \mapsto \varphi(x,y)$ are concave and convex, respectively. In this case, this cost function defines the *skew-Lagrangian* by (1.137),

$$L(x,p) = \sup_y \{\langle y, p \rangle - \varphi(x,y)\},$$

see §1.3.5.

1.2.4. Variational Equivalence

Above mentioned global theory can be localized by sub-differentials. First, given $F : X \to [-\infty, +\infty]$, $x \in X$, and $p \in X^*$, we define $p \in \partial F(x)$ and $x \in \partial F^*(p)$ by

$$F(y) \geq F(x) + \langle y - x, p \rangle, \qquad y \in X$$

and

$$F^*(q) \geq F^*(p) + \langle x, q - p \rangle, \qquad q \in X^*,$$

respectively.
It is obvious that $\partial F(x) \neq \emptyset$ implies $x \in D(F)$, but if $F : X \to (-\infty, +\infty]$ is proper, convex, lower semi-continuous, then

$$x \in \partial F^*(p) \qquad \Leftrightarrow \qquad p \in \partial F(x), \tag{1.143}$$

and Fenchel-Moreau's identity

$$F(x) + F^*(p) = \langle x, p \rangle \tag{1.144}$$

follows, see.[83] If $p \in \partial F(x)$, then $x \in \partial F^*(p)$, and, therefore,

$$F^*(p) - \langle x, p \rangle = -F(x).$$

This relation implies

$$L|_{p\in\partial F(x)} = J, \tag{1.145}$$

recall

$$\begin{aligned} &L(x,p) = F^*(p) + G(x) - \langle x, p \rangle \\ &J(x) = G(x) - F(x) \\ &J^*(p) = F^*(p) - G^*(p). \end{aligned}$$

Similarly, we obtain

$$L|_{x\in\partial G^*(p)} = J^*, \tag{1.146}$$

and thus, unfolding Legendre transformation (1.145)-(1.146) comparable to (1.132), (1.134) arises in this abstract framework. The minimality such as (1.135), on the other hand, follows immediately from (1.142). More precisely, we have

$$\begin{aligned} &L(x,p) \geq J(x) \\ &L(x,p) \geq J^*(p) \end{aligned}$$

and, therefore,

$$L(x,p) \geq \max\{J(x), J^*(p)\} \tag{1.147}$$

for any $(x,p) \in X \times X^*$, and in this way, properties (1.132), (1.134), and (1.135) are obtained in the context of convex analysis.
The first part of our local theory is the following variational equivalence.

Theorem 1.20. *Let $F,G : X \to (-\infty,+\infty]$ be proper, convex, lower semi-continuous, and $L = L(x,p)$ be defined by (1.138). Given $(\overline{x},\overline{p}) \in X \times X^*$, the sets of minimizers of $p \in X^*$ and $x \in X$ of the variational problems*

$$\begin{aligned} J(\overline{x}) &= \inf_{p\in X^*} L(\overline{x},p) \\ J^*(\overline{p}) &= \inf_{x\in X} L(x,\overline{p}) \end{aligned}$$

are denoted by $A^*(\overline{x})$ and $A(\overline{p})$. We say that $\overline{x} \in X$ and $\overline{p} \in X^*$ are critical points of J and J^* if

$$\begin{aligned} &\partial G(\overline{x}) \cap \partial F(\overline{x}) \neq \emptyset \\ &\partial G^*(\overline{p}) \cap \partial F^*(\overline{p}) \neq \emptyset, \end{aligned}$$

and that $(\overline{x},\overline{p})$ is a critical point of L if

$$\begin{aligned} &0 \in \partial_x L(\overline{x},\overline{p}) \\ &0 \in \partial_p L(\overline{x},\overline{p}). \end{aligned}$$

Then, first, we have

$$A^*(x) = \partial F(x)$$
$$A(p) = \partial G^*(p) \tag{1.148}$$

for any $(x,p) \in X \times X^*$. *Next, the following conditions are equivalent.*

(1) $(\overline{x},\overline{p}) \in X \times X^*$ *is a critical point of* L.
(2) $\overline{x} \in X$ *is a critical point of* J *and it holds that*

$$\overline{p} \in \partial G(\overline{x}) \cap \partial F(\overline{x}).$$

(3) $\overline{p} \in X^*$ *is a critical point of* J^* *and it holds that*

$$\overline{x} \in \partial F^*(\overline{p}) \cap \partial G^*(\overline{p}).$$

Finally, we have

$$L(\overline{x},\overline{p}) = J(\overline{x}) = J^*(\overline{p}) \tag{1.149}$$

if one of the above conditions is satisfied.

Proof: By (1.138) and (1.143), it holds that

$$0 \in \partial_x L(x,p) = 0 \Leftrightarrow p \in \partial G(x) \Leftrightarrow x \in \partial G^*(p)$$
$$0 \in \partial_p L(x,p) = 0 \Leftrightarrow x \in \partial F^*(p) \Leftrightarrow p \in \partial F(x) \tag{1.150}$$

for any $(x,p) \in X \times X^*$. Given $x \in X$, we take $p \in A^*(x)$. This p attains

$$J(x) = \inf_{p \in X^*} L(x,p),$$

and, therefore, $0 \in \partial_p L(x,p)$. Thus $A^*(x) = \partial F(x)$ holds by (1.150). The relation $A(p) = \partial G^*(p)$ follows similarly and the first part, (1.148), is proven.
The second part, the equivalence of the above mentioned three conditions, is obtained also by (1.150) because $(\overline{x},\overline{p}) \in X \times X^*$ is a critical point of

$$L(x,p) = F^*(p) + G(x) - \langle x,p \rangle$$

if and only if

$$\overline{p} \in \partial G(\overline{x})$$
$$\overline{x} \in \partial F^*(\overline{p}). \tag{1.151}$$

Finally, (1.149) follows from (1.148) and (1.151). More precisely,

$$\begin{aligned} L(\overline{x},\overline{p}) &= F^*(\overline{p}) + G(\overline{x}) - \langle \overline{x},\overline{p} \rangle \\ &= F^*(\overline{p}) - G^*(\overline{p}) \\ &= G(\overline{x}) - F(\overline{x}). \end{aligned}$$

The proof is complete. □

1.2.5. Spectral Equivalence

Under the assumptions of Theorem 1.20, we have

$$\overline{p} \in \partial G(\overline{x}) \cap \partial F(\overline{x}) \qquad \Leftrightarrow \qquad \overline{x} \in \partial F^*(\overline{p}) \cap \partial G^*(\overline{p}),$$

and, therefore, each critical point of J produces that of J^*, and the converse is also true. This correspondence

$$\overline{x} \quad \leftrightarrow \quad \overline{p}$$

may be called the Legendre transformation of the critical point, and furthermore, $(\overline{x}, \overline{p})$ is a critical point of L in this case. This equivalence of critical points is valid up to their Morse indices, under reasonable assumptions as we are showing. This property is the second part of the local theory of the Toland duality, the spectral equivalence.
Recall that X denotes a Banach space over **R** and

$$F, G : X \to (-\infty, +\infty]$$

are proper, convex, lower-semicontinuous. We define $L(x,p)$, $J(x)$, and $J^*(p)$ by (1.138), (1.139), and (1.140), respectively, and let $(\overline{x}, \overline{p}) \in X \times X^*$ be a critical point of $L = L(x,p)$. Let Y be a C^1 manifold or a closed convex set in X containing $\overline{x}$, and let Y^0 be the tangent hyper-plane of $Y \cap D(J)$ at $\overline{x}$. First, if there is

$$Y_k \subset Y \cap D(J)$$

containing $\overline{x}$ and locally homeomorphic to a closed subspace of Y^0 of codimension k, such that $\overline{x}$ is a minimizer (or strict minimizer) of

$$J|_{Y_k},$$

then we write

$$i_Y(\overline{x}) \leq k \qquad (\text{or} \quad i_Y^s(\overline{x}) \leq k).$$

Next, we call the minimum of such k the (strict) local index of $\overline{x}$ relative to Y and write

$$i_Y(\overline{x}) \qquad (\text{or} \quad i_Y^s(\overline{x})).$$

If there is

$$\check{Y}_k \subset Y \cap D(J)$$

containing $\overline{x}$ and locally homeomorphic to a closed subspace of Y^0 of dimension k, such that $\overline{x}$ is a (strict) maximizer of

$$J|_{\check{Y}_k},$$

then we write

$$\check{i}(\overline{x}) \geq k \qquad (\text{or} \quad \check{i}^s(\overline{x}) \geq k),$$

and the maximum of such k is called the (strict) anti-local index of $\hat{x}$ relative to Y, and is written by

$$\check{i}(\overline{x}) \qquad (\text{or} \quad \check{i}^s(\overline{x})).$$

If Y_* is a C^1 manifold or a closed convex set in X^* containing $\overline{p}$, then the (strict) local index of $\overline{p}$ relative to Y_*, denoted by

$$\check{i}_{Y_*}(\overline{p}) \qquad (\text{or} \quad \check{i}^s_{Y_*}(\overline{p})),$$

is defined similarly, using the tangent hyper-plane Y_*^0 of $Y_* \cap D(J^*)$ at $\overline{p}$.
If $\mathscr{O}_* \subset Y_*$ is an open set containing $\overline{p}$, then the mapping

$$T : \mathscr{O}_* \to Y$$

with $T(\overline{p}) = \overline{x}$ is said to be k-(Y,Y_*) *faithful* if $T^{-1}(Y_k)$ is locally homeomorphic to a closed subspace of Y_*^0 of codimension k containing $\overline{p}$, provided that Y_k is a closed subspace of Y of codimension k containing $\overline{x}$. Similarly, it is anti k-(Y,Y_*) faithful if $T^{-1}(\check{Y}_k)$ is locally homeomorphic to a closed subspace of Y_*^0 of dimension k containing $\overline{p}$, provided that $\check{Y}_k$ is a closed subspace of Y of dimension k containing $\overline{x}$. If $\mathscr{O} \subset Y$ is an open set containing $\overline{x}$, then the (anti) k-(Y_*,Y) faithfulness of the mapping

$$T_* : \mathscr{O} \to Y_*$$

with $T_*(\overline{x}) = \overline{p}$ is defined similarly.

Theorem 1.21. *Under the assumptions of Theorem 1.20, let $(\overline{x},\overline{p})$ be a critical point of $L = L(x,p)$, and let*

$$k = i_Y^{(s)}(\overline{x})$$
$$k_* = i_{Y_*}^{(s)}(\overline{p})$$

be the (strict) local indices of $\overline{x}$ and $\overline{p}$ relative to Y and Y_*, respectively. Then, if

$$k_*\text{-}(Y_*,Y) \textit{ faithful } T_* \subset \partial G$$
$$k\text{-}(Y,Y_*) \textit{ faithful } T \subset \partial F^*,$$

it follows that $k = k_*$, that is

$$i_Y^{(s)}(\overline{x}) = i_Y^{(s)}(\overline{p}).$$

A similar fact holds for the anti-local indices.

Proof: We describe the proof only for the strict indices because the other cases are similar. Since $k = i_Y^s(\overline{x})$, there is $Y_k \subset Y \cap D(J)$ containing $\overline{x}$ and locally homeomorphic to a closed subspace of Y^0 of codimension k such that $\overline{x}$ is a strict minimizer of $J|_{Y_k}$:

$$x \in Y_k \setminus \{\overline{x}\} \Rightarrow J(x) > J(\overline{x}). \tag{1.152}$$

Since $\overline{p} \in A^*(\overline{x})$, we have

$$\begin{aligned} J(\overline{x}) &= \inf_{p \in X^*} L(\overline{x}, p) = L(\overline{x}, \overline{p}) \\ &\geq J^*(\overline{p}) = \inf_{x \in X} L(x, \overline{p}) \end{aligned} \tag{1.153}$$

and also

$$p \in A^*(x) \Leftrightarrow x \in \partial F^*(p).$$

If $p \in T^{-1}(Y_k) \setminus \{\overline{p}\}$, therefore, then

$$Tp \in \partial F^*(p)$$

by $T \subset \partial F^*$, and hence $p \in A^*(Tp)$ which implies that

$$\begin{aligned} J(Tp) &= \inf_{q \in X^*} L(T(p), q) = L(T(p), p) \\ &\leq \inf_{x \in X} L(x, p) = J^*(p). \end{aligned}$$

We have, on the other hand,

$$J(Tp) > J(\overline{x}) \geq J^*(\overline{p})$$

by $Tp \in Y_k$, (1.152), and (1.153), and, therefore, it holds that

$$J^*(p) > J^*(\overline{p})$$

for any $p \in T^{-1}(Y_k) \setminus \{\overline{p}\}$ which means that

$$i^s_{Y_*}(\overline{p}) \leq k = i^s_Y(\overline{x})$$

because T is k-(Y, Y_*) faithful. □

1.2.6. Stability

The stability of the stationary state is derived only from the unfolding- minimality which comprises the third part of the local theory of the Toland duality. Under the notation fo the following theorem, a critical point of C^1-functional can be ”linearly stable“ even when its linearized operator is degenerate.

Theorem 1.22. *Let X be a Banach space over* **R**, *and*

$$F : X \to (-\infty, +\infty]$$

be proper, convex, lower semi-continuous. Let

$$\begin{aligned} J &: X \to [-\infty, +\infty] \\ L &: X \times X^* \to [-\infty, +\infty] \end{aligned}$$

satisfy

$$\begin{aligned} &L|_{p \in \partial F(x)} = J \\ &L(x, p) \geq J(x), \qquad (x, p) \in X \times X^*. \end{aligned}$$

Let Y_0 be a closed subset of a Banach space Y, which is continuously imbedded in X, and Y_* be a Banach space continuously imbedded in X^*, and assume that

$$(\overline{x},\overline{p}) \in D(L) = \{(x,p) \mid L(x,p) \neq \pm\infty\} \subset X \times X^*$$

is in

$$\begin{aligned}&\overline{p} \in \partial F(\overline{x}) \cap Y_*\\&\overline{x} \in Y_0.\end{aligned}$$

Suppose that $\overline{x}$ is a linearly stable local minimizer of

$$J|_{Y_0}$$

in the sense that there is $\varepsilon_0 > 0$ such that any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ satisfying

$$\begin{array}{l}x \in Y_0\\ \|x - \overline{x}\|_Y < \varepsilon_0\\ J(x) - J(\overline{x}) < \delta\end{array} \quad \Rightarrow \quad \|x - \overline{x}\|_Y < \varepsilon, \tag{1.154}$$

and suppose also that $L|_{Y_0 \times Y_*}$ is continuous at $(\overline{x},\overline{p})$. Then, this $(\overline{x},\overline{p})$ is dynamically stable in x component concerning the orbit

$$\{(x(t), p(t))\}_{0 \le t < T} \subset Y_0 \times Y_*,$$

such that

$$t \in [0,T) \mapsto x(t) \in Y_0$$

is continuous and

$$t \in [0,T) \mapsto L(x(t), p(t)) \tag{1.155}$$

is non-increasing which means that any $\varepsilon > 0$ admits $\delta > 0$ such that

$$\begin{aligned}&\|x(0) - \overline{x}\|_Y < \delta\\&\|p(0) - \overline{p}\|_{Y_*} < \delta\end{aligned} \tag{1.156}$$

implies

$$\sup_{t \in [0,T)} \|x(t) - \overline{x}\|_Y < \varepsilon. \tag{1.157}$$

Similarly, if

$$G : X \to (-\infty, +\infty]$$

is proper, convex, lower semi-continuous, if

$$J^* : X^* \to [-\infty, +\infty]$$

satisfies

$$\begin{aligned}&L|_{x \in \partial G^*(p)} = J^*\\&L(x,p) \ge J^*(x), \qquad (x,p) \in X \times X^*,\end{aligned}$$

if $(\overline{x},\overline{p}) \in D(L)$ *is in*

$$\overline{x} \in \partial G^*(\overline{p}) \cap Y$$
$$\overline{p} \in Y_{0*},$$

where Y_{0*} *is a closed subset of* Y_**,* $\overline{p}$ *is a linearly stable local minimizer of*

$$J^*|_{Y_{0*}}$$

in the sense that any $\varepsilon \in (0,\varepsilon_0/4]$ *admits* $\delta > 0$ *such that*

$$\begin{array}{l} p \in Y_{0*} \\ \|p-\overline{p}\|_{Y_*} < \varepsilon_0 \\ J^*(p) - J^*(\overline{p}) < \delta \end{array} \quad \Rightarrow \quad \|p-\overline{p}\|_{Y_*} < \varepsilon,$$

if the orbit $\{(x(t),p(t))\}_{0\leq t<T} \subset Y \times Y_{0*}$ *is given with*

$$t \in [0,T) \mapsto p(t) \in Y_{0^*}$$

continuous and (1.155) decreasing, $\overline{x} \in \partial G^*(\overline{p})$*, and finally if* $L|_{Y\times Y_{0*}}$ *is continuous at* $(\overline{x},\overline{p})$*, then any* $\varepsilon > 0$ *admits* $\delta > 0$ *such that (1.156) implies*

$$\sup_{t\in[0,T)} \|p(t)-\overline{p}\|_{Y_*} < \varepsilon.$$

Proof: We show the former part only. Given $\varepsilon \in (0,\varepsilon_0/4]$, we take δ of (1.154), denoted by $\delta_1 > 0$. Since $L|_{Y_0\times Y_*}$ is continuous at $(\overline{x},\overline{p})$, there is $\delta \in (0,\varepsilon_0/2]$ such that

$$\begin{array}{l} \|x(0)-\overline{x}\|_Y < \delta \\ \|p(0)-\overline{p}\|_{Y_*} < \delta \end{array} \tag{1.158}$$

imply

$$L(x(0),p(0)) - L(\overline{x},\overline{p}) < \delta_1. \tag{1.159}$$

We have, on the other hand,

$$W(x,p) \geq J(x) \geq J(\overline{x}) = W(\overline{x},\overline{p})$$

for any $(x,p) \in Y_0 \times X^*$ with $\|x-\overline{x}\|_Y < \varepsilon_0$ from the assumption. Therefore, as far as

$$\|x(t)-\overline{x}\|_Y < \varepsilon_0 \tag{1.160}$$

it holds that

$$\begin{aligned} 0 &\leq J(x(t)) - J(\overline{x}) \\ &\leq W(x(t),p(t)) - J(\overline{x}) \\ &\leq L(x(0),p(0)) - L(\overline{x},\overline{p}) < \delta_1. \end{aligned} \tag{1.161}$$

Now, we have

$$\|x(0)-\overline{x}\|_Y < \delta \leq \varepsilon_0/2,$$

and if there is $t_0 \in (0,T)$ such that

$$\|x(t_0) - \overline{x}\|_Y = \varepsilon_0/2,$$

then we have (1.160) and hence (1.161) for $t = t_0$ which implies that

$$\|x(t_0) - \overline{x}\|_Y < \varepsilon \le \varepsilon_0/4 \tag{1.162}$$

from (1.154) (with $\delta = \delta_1$), a contradiction.
Since $t \in [0,T) \mapsto x(t) \in Y_0 \subset Y$ is continuous, the relation

$$\|x(t) - \hat{x}\|_Y < \varepsilon_0/2$$

keeps for $t \in [0,T)$, and hence (1.160). Again this implies (1.161) and (1.162) for any $t \in [0,T)$, and the proof is complete. □

1.2.7. Gradient Systems with Duality

This paragraph is devoted to several comments concerning the above abstract results. Let X, Y be Banach spaces with continuous inclusion $Y \subset X$, and Y_* be another Banach space continuously imbedded in X^*. Furthermore, $Y_0 \subset Y$ and $Y_{0*} \subset Y_*$ are closed subsets, and $L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$ be the Lagrangian defined by proper, convex, lower semi-continuous mappings $F, G : X \to (-\infty, +\infty]$.
Given a continuous orbit

$$t \in [0,\infty) \mapsto (x(t), p(t)) \in Y_0 \times Y_{0*},$$

we can define its ω-limit set by

$$\begin{aligned}\omega\,(x(0), p(0)) = \{&(x_\infty, p_\infty) \mid \text{there exists } t_k \to +\infty \text{ such that}\\ &(x(t_k), p(t_k)) \to (x_\infty, p_\infty) \text{ in } Y_0 \times Y_{0*}.\}\end{aligned}$$

In the case that

$$t \mapsto L(x(t), p(t)) \in \mathbf{R}$$

is non-increasing and (x^1, p^1) and (x^2, p^2) are elements in $\omega(x(0), p(0))$, we obtain

$$\begin{aligned} t_k^1 \to +\infty &\quad \text{such that } (x(t_k^1), p(t_k^1)) \to (x^1, p^1) \text{ in } Y_0 \times Y_{0*}\\ t_k^2 \to +\infty &\quad \text{such that } (x(t_k^2), p(t_k^2)) \to (x^2, p^2) \text{ in } Y_0 \times Y_{0*}\end{aligned}$$

such that $t_k^1 < t_k^2 < t_{k+1}^1$ $(k = 1, 2, \ldots)$ which implies

$$L(x(t_k^1), p(t_k^1)) \ge L(x(t_k^2), p(t_k^2)) \ge L(x(t_{k+1}^1), p(t_{k+1}^1)),$$

and, therefore,

$$L(x^1, p^1) = L(x^2, p^2),$$

provided that

$$L : Y_0 \times Y_{0*} \to \mathbf{R}$$

is continuous. Thus L is invariant on $\omega(x(0), p(0))$.
If there is a continuous local semi-flow $\{T_t\}$ on $Y_0 \times Y_{0*}$ such that

$$(x(t), p(t)) = T_t(x(0), p(0)),$$

then we obtain

$$T_t(x_\infty, p_\infty) \in \omega(x(0), p(0))$$

for any $(x_\infty, p_\infty) \in \omega(x(0), p(0))$, and, therefore,

$$L(T_t(x_\infty, p_\infty)) = \text{constant} \tag{1.163}$$

for any $t \geq 0$. This relation means $\omega(x(0), p(0)) \subset E$, where E denotes the set of $(x_\infty, p_\infty) \in Y_0 \times Y_{0*}$ satisfying (1.163), which may be called the *stationary set*. The orbit

$$\mathscr{O} = \{(x(t), p(t)) \mid 0 \leq t < +\infty\}$$

is continuous in $Y_0 \times Y_{0*}$, and, therefore, the ω-limit set $\omega(x(0), p(0))$ is connected and compact, provided that $\mathscr{O}$ is compact.[138] Under the assumption of Theorem 1.22, on the other hand, the critical point $(\overline{x}, \overline{p})$ of L is isolated. Therefore, if the above semi-flow $\{T_t\}$ is global in time around it, then it is asymptotically stable and it holds that

$$\lim_{t \to +\infty} \|x(t) - \overline{x}\|_Y = \lim_{t \to +\infty} \|p(t) - \overline{p}\|_{Y_*} = 0. \tag{1.164}$$

The Lagrangian

$$L(x, p) = F^*(p) + G(x) - \langle x, p \rangle$$

induces the gradient system, say

$$\begin{aligned} \dot{p} &\in -L_p(x, p) \\ \tau \dot{x} &\in -L_x(x, p), \end{aligned} \tag{1.165}$$

using the combination of model (A) - model (A) equations, see §1.3.1, where $\tau > 0$ is the relaxization time. Then, its simplified system is defined by

$$\begin{aligned} \dot{p} &\in -L_p(x, p) \\ 0 &\in L_x(x, p) \end{aligned}$$

or

$$\begin{aligned} 0 &\in L_p(x, p) \\ \tau \dot{x} &\in -L_x(x, p). \end{aligned}$$

Equation (1.165) means

$$\begin{aligned} F^*(q) - F^*(p) &\geq \langle x - \dot{p}, q - p \rangle \\ G(y) - G(x) &\geq \langle y - x, p - \tau \dot{x} \rangle \end{aligned}$$

for any $(y, q) \in X \times X^*$. More precisely, assuming a Hilbert space Y over $\mathbf{R}$ with the continuous inclusion

$$X \subset Y \approx Y^* \subset X^*, \tag{1.166}$$

we define the solution $(p,x) = (p(t),x(t))$ to (1.165) by

$$p \in C([0,T),Y), \ \ \dot{p} \in L^\infty_{\rm loc}([0,T),Y)$$
$$x \in C([0,T),Y), \ \ \dot{x} \in L^\infty_{\rm loc}([0,T),Y). \tag{1.167}$$

Then, it holds that

$$\limsup_{h\downarrow 0} \frac{1}{h} \{F^*(p(t)) - F^*(p(t-h))\} \leq \langle x(t), \dot{p}(t)\rangle - \|\dot{p}(t)\|_Y^2$$
$$\limsup_{h\downarrow 0} \frac{1}{h} \{G(x(t)) - G(x(t-h))\} \leq \langle \dot{x}(t), p(t)\rangle - \tau \|\dot{x}(t)\|_Y^2$$

for $t \in (0,T)$, and, therefore,

$$\begin{aligned} & p(t) \in D(F^*) \\ & x(t) \in D(G), \qquad 0 \leq t < T \\ & \Rightarrow \\ & p(T) \in D(F^*), \ \ x(T) \in D(G), \\ & L(x(T),p(T)) \leq \liminf_{t\uparrow T} L(x(t),p(t)). \end{aligned}$$

Thus, if $(\overline{x},\overline{p}) \in Y \times Y$ is a linearly stable critical point of L, and (1.165) is locally well-posed in a neighborhood of $(\overline{x},\overline{p})$ in the sense of (1.167), then it is globally well-posed there, and $(\overline{x},\overline{p})$ is dynamically stable.

1.2.8. Entropy Functional

We apply the above formulated abstract theory to Helmholtz' free energy defined in §1.1.3, where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. This paragraph is devoted to a convex analytic approach to what is written in Chapter 9 of.[304]
Given a positive definite self-adjoint operator A in $L^2(\Omega)$ with compact resolvent, we take the Gel'fand triple[161,314]

$$X \hookrightarrow L^2(\Omega) \approx L^2(\Omega)^* \hookrightarrow X^*$$

for $X = D(A^{1/2})$, and define the *dual entropy functional* $F : X \to (-\infty, +\infty]$ by

$$F(v) = \lambda \log\left(\int_\Omega Ve^v\right) + \lambda(\log\lambda - 1),$$

where $V = V(x) > 0$ is a continuous function of $x \in \overline{\Omega}$. This functional is proper, convex, lower semi-continuous, and it holds that

$$D(F) = \left\{ v \in X \mid \int_\Omega Ve^v < +\infty \right\}$$
$$u \in \partial F(v) \quad \Leftrightarrow \quad u = \frac{\lambda Ve^v}{\int_\Omega Ve^v}.$$

The *entropy functional* $F^*: X^* \to (-\infty, +\infty]$ is defined by the second Legendre transformation of the dual entropy functional,

$$F^*(u) = \sup_{v \in X} \{\langle v, u \rangle - F(v)\},$$

and it holds that

$$F^*(u) = \begin{cases} \int_\Omega u(\log u - 1 - \log V), & u \in D(F^*) \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$D(F^*) = \{u \in X^* \mid u \geq 0,\ u \in (L\log L)(\Omega),\ \|u\|_1 = \lambda\}$$

with $(L\log L)(\Omega)$ denoting the Zygmund space, see §1.2.9. Thus $v \in \partial F^*(u)$ if and only if $u \in D(F^*)$ and

$$v = \log u - \log V + \text{constant} \in X.$$

Putting

$$G(v) = \frac{1}{2}\left\|A^{1/2}v\right\|_2^2,$$

next, we obtain a proper, convex, lower semi-continuous mapping

$$G: X \to (-\infty, +\infty).$$

The self-adjoint operator A induces the isomorphism $\hat{A}: X \to X^*$, and we have

$$G^*(u) = \frac{1}{2}\langle \hat{A}^{-1}u, u\rangle$$

for $u \in X^*$. Then, the Lagrangian is defined by

$$L(u,v) = F^*(u) + G(v) - \langle v, u\rangle, \tag{1.168}$$

with the stationary state described by

$$\begin{aligned} 0 &\in L_u(\overline{u}, \overline{v}) \\ 0 &\in L_v(\overline{u}, \overline{v}) \end{aligned}$$

or, equivalently,

$$\begin{aligned} \overline{u} &= \hat{A}^{-1}\overline{v} \\ \overline{v} &\in \partial F^*(\overline{u}). \end{aligned}$$

Here, we adopt the notation $L(u,v)$, following (1.79).
From Theorem 1.20, this relation splits into the conditions on $\overline{u}$ and $\overline{v}$, that is, to be a critical point of

$$\begin{aligned} J(v) &= G(v) - F(v) \\ &= \frac{1}{2}\left\|A^{1/2}v\right\|_2^2 - \lambda\log\left(\int_\Omega Ve^v\right) + \lambda(\log\lambda - 1) \end{aligned}$$

defined for

$$v \in X, \qquad \int_\Omega Ve^v < +\infty,$$

and that of

$$\begin{aligned} J^*(u) &= F^*(u) - G^*(u) \\ &= \int_\Omega u(\log u - 1 - \log V) - \frac{1}{2}\left\langle \hat{A}^{-1}u, u \right\rangle \end{aligned}$$

defined for

$$u \in X^* \cap L^1(\Omega), \quad u \geq 0, \quad \|u\|_1 = \lambda.$$

These variational problems are equivalent to

$$\overline{v} \in X, \qquad \int_\Omega Ve^{\overline{v}} < +\infty, \qquad \hat{A}\overline{v} = \frac{\lambda Ve^{\overline{v}}}{\int_\Omega Ve^{\overline{v}}} \in X^* \tag{1.169}$$

and

$$\begin{aligned} &\overline{u} \in X \cap (L\log L)(\Omega), \quad \overline{u} \geq 0, \quad \|\overline{u}\|_1 = \lambda \\ &\hat{A}^{-1}\overline{u} = \log \overline{u} - \log V + \text{constant} \in X, \end{aligned} \tag{1.170}$$

respectively.
The condition

$$\left.\begin{array}{l} v \in L^2(\Omega) \\ \int_\Omega v = 0 \\ A^{-1}v = \text{constant} \end{array}\right. \quad \Rightarrow \quad v = 0$$

is satisfied in many cases. Assuming this property, we can define (Y, Y_*)-faithful $T \subset \partial F^*$ for $Y = D(J)/\mathbf{R}$ or $Y = D(J)$ and $Y_* = D(J^*)$, namely, $v = Tu$ if and only if

$$v = \log u - \log V + \text{constant} \in X,$$

or, equivalently,

$$u = \frac{\lambda Ve^v}{\int_\Omega Ve^v}.$$

This situation is the same as that of Chapter 9 of[304] and details are omitted.
The Fréchet derivative

$$dG = \hat{A} : X \to X^*,$$

next, is an isomorphism, and hence

$$T = dG|_Y : Y \to Y_*$$

is also faithful by the duality $EXP'(\Omega) \approx L\log L(\Omega)$, see[1,264] and §1.2.9 for these function spaces. Thus, we obtain the variational and spectral equivalences of these J and J^*, together with the unfolding-minimality using the Lagrangian $L = L(u,v)$ defined by (1.168).

If $(\overline{u},\overline{v})$ is a linearly stable critical point of L, then it is dynamically stable for

$$\begin{aligned} u_t &= \nabla\cdot(u\nabla L_u(u,v)) \\ \tau v_t &= -L_v(u,v) \end{aligned} \tag{1.171}$$

in the sense that any $\varepsilon > 0$ admits $\delta > 0$ such that

$$\begin{aligned} &\|v(\cdot,0)-\overline{v}\|_X < \delta \\ &\|u(\cdot,0)-\overline{u}\|_{X^*\cap L\log L} < \delta \\ &\|u(\cdot,0)\|_1 = \lambda = \|\overline{u}\|_1 \end{aligned}$$

implies

$$\begin{aligned} &\sup_{t\in[0,T)} \|v(\cdot,t)-\overline{v}\|_X < \varepsilon \\ &\sup_{t\in[0,T)} \|u(\cdot,t)-\overline{u}\|_{X^*\cap L\log L} < \varepsilon. \end{aligned}$$

This dynamical stability is actually the case if the solution is sufficiently regular, say,

$$t \mapsto (u(\cdot,t),v(\cdot,t)) \in D(F^*)\times X$$

is continuous,

$$t \mapsto L(u(\cdot,t),v(\cdot,t))$$

is non-increasing, and $\|u(\cdot,t)\|_1$ is invariant in t, see Theorem 1.22.
If $\overline{v}$ is slightly regular, say, $e^{\overline{v}} \in L^p(\Omega)$ for $p > 1$, then this critical point $(\overline{u},\overline{v})$ of $L = L(u,v)$ comprises the classical solutions to (1.169) and (1.170). In this case, which is always true if $n = 2$, the functionals

$$\begin{aligned} J^*(u) &= \int_\Omega u(\log u - 1 - \log V) - \frac{1}{2}\left\langle A^{-1}u,u\right\rangle, \qquad u \geq 0,\ \|u\|_1 = 1 \\ J(v) &= \frac{1}{2}\left\|A^{1/2}v\right\|_2^2 - \lambda\log\left(\int_\Omega Ve^v\right) + \lambda\log\lambda - \lambda \end{aligned}$$

are twice differentiable at $u = \overline{u}$, $v = \overline{v}$ which results in the derivation of Morse indices by linearized operators and which are equal to the strict local indices and the strict local anti-indices of the critical point $(\overline{u},\overline{v})$ of

$$L(u,v) = \frac{1}{2}\left\|A^{1/2}v\right\|_2^2 + \int_\Omega u(\log u - 1 - \log V) - \langle v,u\rangle.$$

More precisely, the linearized operator derived from above $J = J(v)$ and $v = \overline{v}$ is defined by

$$\mathscr{L}\psi = A\psi - \lambda\left(\frac{Ve^{\overline{v}}\psi}{\int_\Omega Ve^{\overline{v}}} - \frac{\int_\Omega Ve^{\overline{v}}\psi}{(\int_\Omega Ve^{\overline{v}})^2}\cdot Ve^{\overline{v}}\right)$$

with the domain $D(\mathscr{L}) = D(A)$ in $L^2(\Omega)$, and then, the non-local term is eliminated by the transformation

$$\varphi = \psi - \frac{\int_\Omega Ve^{\overline{v}}\psi}{\int_\Omega Ve^{\overline{v}}}.$$

Thus if $v = Au$ means (1.15), then we obtain the eigenvalue problem

$$-\Delta\varphi = \mu\overline{u}\varphi \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{1.172}$$

for

$$\overline{u} = \frac{\lambda V e^{\overline{v}}}{\int_\Omega V e^{\overline{v}}}, \tag{1.173}$$

and the (strict) local index of $\overline{v}$ and hence that of $\overline{u}$ are equal to the number of eigenvalues in $0 < \mu \leq 1$ $(0 < \mu < 1)$ defined by (1.172).
Similarly, if $v = Au$ is defined by

$$-\Delta v = u \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then the (strict) local index of $\overline{v}$ and hence that of $\overline{u}$ are equal to the number of eigenvalues in $0 < \mu \leq 1$ $(0 < \mu < 1)$ of (1.123), that is

$$-\Delta\varphi = \mu\overline{u}\varphi \quad \text{in } \Omega, \qquad \varphi = \text{constant on } \partial\Omega$$
$$\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} = 0 \tag{1.174}$$

for $\overline{u} = \overline{u}(x)$ defined by (1.173), see also.[304]

1.2.9. Zygmund Space

The above mentioned dynamical stability to (1.171) requires the continuity of the entropy functional defined on the Zygmund space.[302] In fact, given an open set $\Omega \subset \mathbf{R}^n$, we define the Zygmund space by

$$(L\log L)(\Omega) = \left\{ f : \Omega \to \Omega \text{ measurable} \mid \|f\|_{L\log L} < +\infty \right\},$$

where

$$\|f\|_{L\log L} = \inf\left\{ k > 0 \mid \int_\Omega |f| \log\left(e + \frac{|f|}{k}\right) \leq k \right\}$$

denotes the Luxemburg norm, see.[1] Then, we obtain the following theorem.[156]

Theorem 1.23. *There is an order-preserving norm in $(L\log L)(\Omega)$ equivalent to the Luxemburg norm defined by*

$$[f]_{L\log L} = \int_\Omega |f(x)| \log\left(e + \frac{|f(x)|}{\|f\|_1}\right),$$

under the agreement that $[f]_{L\log L} = 0$ if $\|f\|_1 = 0$.

The last paragraph of this section is devoted to the proof of Theorem 1.23 based on a series of lemmas.

Lemma 1.1. *It holds that*

$$f\log\left(e + \frac{f}{a}\right) - \frac{fb}{a-b}\log\frac{a}{b} \geq g\log\left(e + \frac{g}{a}\right) - \frac{gb}{a-b}\log\frac{a}{b} \tag{1.175}$$

for $f \geq g \geq 0$ *and* $a > b > 0$.

Proof: We have

$$\log x \leq x - 1, \qquad x > 0$$

and, therefore,

$$b \log \frac{a}{b} \leq a - b, \qquad a > b > 0.$$

Then, putting

$$h(s) = s \log\left(e + \frac{s}{a}\right) - \frac{sb}{a-b} \log \frac{a}{b},$$

we obtain

$$\begin{aligned} h'(s) &= \log\left(e + \frac{s}{a}\right) + s \cdot \frac{1}{e + \frac{s}{a}} \cdot \frac{1}{a} - \frac{b}{a-b} \log \frac{a}{b} \\ &\geq 1 - 1 = 0 \end{aligned}$$

and hence (1.175) follows from $h(f) \geq h(g)$. □

Lemma 1.2. *If* $f, g \in L \log L(\Omega)$ *satisfies*

$$|g(x)| \leq |f(x)| \qquad a.e.,$$

then it holds that

$$[g]_{L \log L} \leq [f]_{L \log L}.$$

Proof: We may suppose that $0 < \|g\|_1 < \|f\|_1$. Then, we obtain

$$\begin{aligned} |g(x)| \log\left(e + \frac{|g(x)|}{\|g\|_1}\right) &\leq |f(x)| \log\left(e + \frac{|f(x)|}{\|f\|_1}\right) \\ &+ \{|g(x)|\, \|f\|_1 - |f(x)|\, \|g\|_1\} \cdot \frac{\log \|f\|_1 - \log \|g\|_1}{\|f\|_1 - \|g\|_1} \end{aligned}$$

and, therefore,

$$\int_\Omega |g(x)| \log\left(e + \frac{|g(x)|}{\|g\|_1}\right) \leq \int_\Omega |f(x)| \log\left(e + \frac{|f(x)|}{\|f\|_1}\right),$$

and the proof is complete. □

Lemma 1.3. *We have*

$$(x+y) \log\left(e + \frac{x+y}{a+b}\right) \leq x \log\left(e + \frac{x}{a}\right) + y \log\left(e + \frac{y}{b}\right), \tag{1.176}$$

where $x, y \geq 0$ *and* $a, b > 0$.

Proof: The function $h(s) = \log\left(e + s^{-1}\right)$ is convex in $s > 0$, and hence it holds that

$$h\left(\frac{a+b}{x+y}\right) \leq \frac{x}{x+y} h\left(\frac{a}{x}\right) + \frac{y}{x+y} h\left(\frac{b}{y}\right).$$

This inequality implies (1.176). □

Lemma 1.4. *The triangle inequality*

$$[f+g]_{L\log L} \leq [f]_{L\log L} + [g]_{L\log L}$$

holds true.

Proof: We have

$$[f+g]_{L\log L} \leq [|f|+|g|]_{L\log L}$$

by Lemma 1.2, and, therefore, may assume $f, g \geq 0$. Then, Lemma 1.3 guarantees

$$\begin{aligned}
[f+g]_{L\log L} &= \int_\Omega (f+g) \log\left(e + \frac{f+g}{\|f+g\|_1}\right) \\
&\leq \int_\Omega \left\{ f \log\left(e + \frac{f}{\|f\|_1}\right) + g \log\left(e + \frac{g}{\|g\|_1}\right) \right\} \\
&= [f]_{L\log L} + [g]_{L\log L},
\end{aligned}$$

and the proof is complete. □

Lemma 1.5. *The norm* $[\,\cdot\,]_{L\log L}$ *is equivalent to the Luxemburg norm, and it holds that*

$$\|f\|_1 \leq \|f\|_{L\log L} \leq [f]_{L\log L} \leq 2\,\|f\|_{L\log L}.$$

Proof: From the definition of the Luxemburg norm, we obtain

$$K = \int_\Omega |f| \log\left(e + \frac{|f|}{K}\right) \geq \|f\|_1$$

for $K = \|f\|_{L\log L}$, which implies

$$K \leq \int_\Omega |f| \log\left(e + \frac{|f|}{\|f\|_1}\right) = [f]_{L\log L}.$$

We have, on the other hand,

$$\begin{aligned}
\int_\Omega |f| \log\left(e + \frac{|f|}{\|f\|_1}\right) &\leq \int_\Omega |f| \log\left(e \cdot \frac{K}{\|f\|_1} + \frac{|f|}{\|f\|_1}\right) \\
&= \int_\Omega |f| \log\left(e + \frac{|f|}{K}\right) + \int_\Omega |f| \log \frac{K}{\|f\|_1} \\
&= K + \|f\|_1 \cdot \log \frac{K}{\|f\|_1},
\end{aligned}$$

and here, the right-hand side is estimated from above by

$$K + \frac{K}{e} \leq 2K = 2\,[f]_{L\log L}$$

because

$$\log s \leq \frac{s}{e} \qquad (s \geq 1).$$

The proof is complete. □

We turn to the following theorem proven by a series of lemmas.[156]

Theorem 1.24. *If Ω is bounded, then the mapping*

$$f \in (L\log L)(\Omega) \mapsto \int_{\Omega} f \log |f| \in \mathbf{R}$$

is well-defined and continuous.

Lemma 1.6. *If x, y are elements in a normed space and $t \in \mathbf{R}$, then it holds that*

$$\left| x\,|x|^{it} + y\,|y|^{it} - (x+y)\,|x+y|^{it} \right| \leq 2\,|t|\,|x+y| \log\left(e + \frac{|x|+|y|}{|x+y|} \right).$$

Proof: For $a \geq b > 0$ and $t \in \mathbf{R}$, we have

$$\left| e^{it} - 1 \right| \leq |t|$$

and hence it holds that

$$\begin{aligned} \left| a^{it} - b^{it} \right| &= \left| \left(\frac{a}{b} \right)^{it} - 1 \right| = \left| e^{it \log \frac{a}{b}} - 1 \right| \\ &\leq |t| \log \frac{a}{b} \leq |t| \left(\frac{a}{b} - 1 \right). \end{aligned}$$

From the symmetry and the homogenity, we may assume

$$|x| + |y| = 1, \qquad 0 < |x| \leq \frac{1}{2} \leq |y|.$$

Then, it follows that

$$\begin{aligned} &|x+y| \leq |x| + |y| = 1 \\ &|y| \leq |x| + |y| = 1, \end{aligned}$$

and we obtain

$$\begin{aligned} &\left| x\,|x|^{it} + y\,|y|^{it} - (x+y)\,|x+y|^{it} \right| \\ &= \left| (x+y) \left(1 - |x+y|^{it} \right) - (x+y) \left(1 - |y|^{it} \right) - x \left(|y|^{it} - |x|^{it} \right) \right| \\ &\leq |t| \left\{ |x+y| \log \frac{1}{|x+y|} + |x+y| \log \frac{1}{|y|} + |x| \cdot \left(\frac{|y|}{|x|} - 1 \right) \right\}. \end{aligned}$$

Here, we have

$$|y| - |x| \leq |y+x|,$$

and the right-hand side is equal to

$$
\begin{aligned}
&|t|\,\{-|x+y|\log|x+y| - |x+y|\log|y| + |y| - |x|\} \\
&\le |t|\left(|x+y|\log\frac{1}{|x+y|\cdot|y|} + |x+y|\right) \\
&= |t|\,|x+y|\log\frac{e}{|x+y|\cdot|y|} \le |t|\,|x+y|\log\frac{2e}{|x+y|} \\
&\le |t|\,|x+y|\log\left(e+\frac{1}{|x+y|}\right)^2 = 2\,|t|\,|x+y|\log\left(e+\frac{1}{|x+y|}\right).
\end{aligned}
$$

The proof is complete. □

Lemma 1.7. *If x, y are elements in a normed space, then it holds that*

$$
\begin{aligned}
&|x\log|x| + |y|\log|y| - (x+y)\log|x+y|| \\
&\le 2\,|x+y|\log\left(e+\frac{|x|+|y|}{|x+y|}\right).
\end{aligned}
$$

Proof: We obtain the result by dividing both sides of inequality in the previous lemma by $|t|$ and making $t \to 0$. □

Lemma 1.8. *It holds that*

$$
\int_\Omega |f-g|\log\left(e+\frac{|f|+|g|}{|f-g|}\right) \le \|f-g\|_1 \log\left(e+\frac{\|f\|_1+\|g\|_1}{\|f-g\|_1}\right),
$$

where $f, g \in L^1(\Omega)$.

Proof: Both sides are zero if $f = g$ a.e. by the definition, and, therefore, we can assume $\|f-g\|_1 = 1$. Thus

$$
\mu(dx) = |f(x)-g(x)|\,dx
$$

is a probability measure while $h(s) = \log(e+s)$ is a concave function of $s > 0$. Jensen's inequality guarantees

$$
\begin{aligned}
\int_\Omega \log\left(e+\frac{|f|+|g|}{|f-g|}\right) d\mu &\le \log\left(e+\int_\Omega \frac{|f|+|g|}{|f-g|} d\mu\right) \\
&= \log\left(e+\int_\Omega |f|+|g|\ dx\right) \\
&= \log(e+\|f\|_1+\|g\|_1),
\end{aligned}
$$

and the proof is complete. □

Lemma 1.9. *It holds that*

$$
\begin{aligned}
&\left|\int_\Omega f\log|f| - g\log|g|\ d\mu\right| \le \|f-g\|_1\,|\log\|f-g\|_1| \\
&+2e^{-1}\,|\Omega|^{1/2}\,\|f-g\|_1 + 2\,\|f-g\|_1 \log\left(e+\frac{\|f\|_1+\|g\|_1}{\|f-g\|_1}\right) \\
&+[f-g]_{L\log L}, \qquad (1.177)
\end{aligned}
$$

where $f,g \in L\log L(\Omega)$.

Proof: We obtain

$$\left| f\log|f| - g\log|g| - (f-g)\log|f-g| \right| \leq 2\,|f-g|\log\left(e+\frac{|f|+|g|}{|f-g|}\right) \tag{1.178}$$

by Lemma 1.7. Then we have

$$\begin{aligned}&\left|\int_\Omega |f-g|\log|f-g|\right| \leq \|f-g\|_1 \left|\int_\Omega \frac{|f-g|}{\|f-g\|_1}\log\frac{|f-g|}{\|f-g\|_1}\right| \\ &+\|f-g\|_1\,|\log\|f-g\|_1|\end{aligned}$$

and use

$$|s\log s| \leq \begin{cases} s\log(s+e) & (s\geq 1) \\ 2e^{-1}s^{1/2} & (0<s<1)\end{cases}$$

for the first term of the right-hand side:

$$\begin{aligned}&\left|\int_\Omega \frac{|f-g|}{\|f-g\|_1}\log\frac{|f-g|}{\|f-g\|_1}\right| \\ &\leq \frac{1}{\|f-g\|_1}\int_\Omega |f-g|\log\left(e+\frac{|f-g|}{\|f-g\|_1}\right) \\ &+2e^{-1}\,|\Omega|^{1/2} = \frac{[f-g]_{L\log L}}{\|f-g\|_1}+2e^{-1}\,|\Omega|^{1/2}.\end{aligned}$$

Now, we apply Lemma 1.8 for the right-hand side of (1.178), and obtain (1.177). The proof is complete. □

1.2.10. Summary

We have described the fundamental structure of dual variation, particularly, the Toland duality.

(1) In the gradient system associated with the Toland duality, stationary states are equivalently formulated variations, in terms of the field and the particle components.
(2) These structures are packaged into the Lagrangian. Then, a linearly stable stationary solution is dynamically stable. This stability splits in both components, field distribution and particle density.
(3) A typical example of the Toland duality is the full system of chemotaxis, associated with the entropy functional, which is continuous on the Zygmund space $(L\log L)(\Omega)$.

1.3. Phenomenology

An important feature of dual variation is the unfolding-minimality concerning the stationary state. There are, however, systems provided only with the ”semi“-duality, that is semi-unfolding and minimality. Among them are the problems arising in phase transition, phase separation, and shape memory alloys. In the Kuhn-Tucker duality, on the other hand, the stable critical state is realized as a saddle. We have several such systems in mathematical biology, game theory, and linear programing. This section is devoted to these variants of the Toland duality.

1.3.1. Non-Convex Evolution

The Ginzburg-Landau theory is a phenomenology consistent to thermodynamics where free energy is defined as a functional of an order parameter, composed of van der Waals' penality and the double-well potential. This free energy is not formulated by the Toland duality because the double-well potential is neither convex nor concave.

Generally, non-equilibrium mean field equations in phenomenological theories are described by the chemical potential

$$\mu = \delta\mathscr{F}(\varphi),$$

where $\mathscr{F} = \mathscr{F}(\varphi)$ stands for the free energy $\mathscr{F} = \mathscr{F}(\varphi)$ defined by the order parameter $\varphi = \varphi(x)$. These equations are classified into model (A), model (B), and model (C) equations.[130,146] In more details, if $\Omega \subset \mathbf{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary $\partial\Omega$, then the order parameter $\varphi = \varphi(x,t)$ is a function of the position $x \in \Omega$ and the time $t > 0$ indicating the status of the material, and $\mathscr{F} = \mathscr{F}(\varphi)$ is a quantity determined by φ. Thus $\mathscr{F} = \mathscr{F}(\varphi)$ is regarded as a functional of $\varphi = \varphi(x,t)$, and the system moves toward the equilibrium, making $\mathscr{F}(\varphi)$ decrease. Here, the chemical potential, $\delta\mathscr{F}(\varphi)$ is defined by

$$\langle \psi, \delta\mathscr{F}(\varphi)\rangle = \frac{d}{ds}\mathscr{F}(\varphi + s\psi)\Big|_{s=0} \tag{1.179}$$

similarly to (1.18). As in the previous cases, this $\langle\ ,\ \rangle$ is usually identified with the L^2 inner product.

Model (A) equation is formulated by

$$\varphi_t = -K\delta\mathscr{F}(\varphi) \qquad \text{in } \Omega \times (0,T),$$

where K is a positive quantity, possibly associated with φ. Then, it holds that

$$\frac{d}{dt}\mathscr{F}(\varphi) = -\int_\Omega K\delta\mathscr{F}(\varphi)^2 \le 0.$$

Model (B) equation, on the other hand, is described by

$$\begin{aligned} &\varphi_t = \nabla\cdot(K\nabla\delta\mathscr{F}(\varphi)) && \text{in } \Omega\times(0,T) \\ &K\frac{\partial}{\partial\nu}\delta\mathscr{F}(\varphi) = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned}$$

In this case, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \varphi = \int_{\partial\Omega} K\frac{\partial}{\partial\nu}\delta\mathscr{F}(\varphi) = 0 \\ &\frac{d}{dt}\mathscr{F}(\varphi) = -\int_\Omega K\,|\nabla\delta\mathscr{F}(\varphi)|^2 \le 0. \end{aligned}$$

Thus, model (A) and model (B) equations stand for the thermodynamically closed system and thermodynamically-materially closed system, respectively. A non-trivial time periodic solution, for example, is not permitted to both of them.
The stationary state is actually defined by the zero free energy consumption; it is

$$\delta\mathscr{F}(\varphi) = 0$$

in the model (A) equation, while

$$\delta\mathscr{F}(\varphi) = 0 \quad \text{constrained by} \quad \int_\Omega \varphi = \lambda$$

in the model (B) equation, where λ is a prescribed constant indentified with an eigenvalue. More precisely, stationary states of model (A) and model (B) equations are defined by

$$\frac{d}{ds}\mathscr{F}(\varphi + s\psi)\bigg|_{s=0} = 0 \qquad \text{for all } \psi$$

and

$$\frac{d}{ds}\mathscr{F}(\varphi + s\psi)\bigg|_{s=0} = 0 \qquad \text{fo all } \psi \text{ with } \textstyle\int_\Omega \psi = 0$$
$$\int_\Omega \varphi = \lambda,$$

respectively. Similarly, the linearized stability of the stationary state φ means

$$Q(\psi,\psi) \equiv \frac{1}{2}\frac{d^2}{ds^2}\mathscr{F}(\varphi + s\psi)\bigg|_{s=0} > 0 \qquad \text{for all } \psi \neq 0$$

in the model (A) equation, while

$$Q(\psi,\psi) > 0 \qquad \text{for all } \psi \neq 0 \text{ with } \textstyle\int_\Omega \psi = 0$$

in the model (B) equation.
If model (A) equation or model (B) equation is well-posed globally in time, the associated semi-flow is completely continuous, and the set of stationary solutions, denoted by E, is bounded in a suitable Banach space, then there is a global *attractor* denoted by $\mathscr{A}$. This $\mathscr{A}$ is connected, comprises of the unstable manifold of E (and of the union of the unstable manifolds of the elements in E if any of them is hyperbolic), and attracts the orbit as $t \uparrow +\infty$ uniformly in the initial value contained in a bounded set, see.[128,227] We hereby describe several free energies and model equations derived from them.

Helmholtz' Free Energy

As we have mentioned in §1.1.3, *Helmholtz' free energy*

$$\mathscr{F}(u) = \alpha \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx'$$

induces the mean field equation of many self-gravitating particles, where $u = u(x,t)$ denotes the particle density. If the absolute temperature α is equal to 1, and the potential $G = G(x,x')$ is the Green's function to (1.15), then we obtain the simplified system of chemotaxis (1.8) as a model (B) equation (1.19) with $K = u$. Its stationary state is described by

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$
$$\int_\Omega v = 0,$$

using the field component, and in two space dimension, the quantized blowup mechanism of this state implies that of the non-equilibrium, see §1.1.2.

Allen-Cahn Equation

Ginzburg-Landau's free energy,

$$\mathscr{F}(\varphi) = \int_\Omega \frac{\xi^2}{2} |\nabla\varphi|^2 + W(\varphi)\, dx \tag{1.180}$$

induces the Allen-Cahn equation[4]

$$\varphi_t = K(\xi^2 \Delta\varphi - W'(\varphi)) \qquad \text{in } \Omega \times (0,T) \tag{1.181}$$

in phase separation as a model (A) equation, where $\xi > 0$ is a constant associated with the intermolecular force, $\varphi = \varphi(x,t)$ is the order parameter, $K > 0$ is a constant, and

$$W(\varphi) = \frac{1}{4}\left(\varphi^2 - 1\right)^2$$

is the double-well potential. Usually, this $\mathscr{F}(\varphi)$ is associated with all $\varphi \in H^1(\Omega)$, and then the natural boundary condition

$$\frac{\partial \varphi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{1.182}$$

is furthermore imposed to (1.181). Thus, the Allen-Cahn equation, (1.181)-(1.182), is formulated by a semilinear parabolic equation of the second order.
Then, the stationary state is described by

$$-\xi^2 \Delta\varphi = \varphi - \varphi^3 \quad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

and its stability is equivalent to the positivity of the first eigenvalue of the self-adjoint operator in $L^2(\Omega)$,

$$A = -\xi^2 \Delta - 1 + 3\varphi^2,$$

with the domain

$$D(A) = \left\{ \psi \in H^2(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}.$$

Here, any non-constant stationary solution φ is linearly unstable if Ω is convex. Actually, this property is the case of the general (single) semilinear parabolic equation with the Neumann boundary condition.[44,199]

In the one-space dimensional case, furthermore, each ω-limit set, see §1.2.7, is composed of one element.[198,354] Actually, due to the parabolic Liouville property,[10,109,200] and the dynamics of the semi-linear parabolic equation in one-space dimension has several remarkable profiles. For example, there are the Morse-Smale property,[9,139] the meandric permutation concerning the stationary solution,[107] and the hetero-clinic cascade.[95,343] The other aspect of such equations is the strong order preserving property. It is derived also from the strong maximum principle and is valid even to the multi-space dimension.[145,201,287] Related aspects of the dynamics of the semilinear parabolic equations are described by[129,148,185] including the multi-space dimensional case.

The first term of the integrand of $\mathscr{F}(\varphi)$ of (1.180) is written as

$$\xi^2 |\nabla\varphi|^2 = \left| \frac{\partial \varphi}{\partial(\xi^{-1}x)} \right|^2.$$

It is associated with the surface tension and is called van der Waals' penalty. Here, the parameter $0 < \xi \ll 1$ is determined in accordance with the molecular distance. The term $W(\varphi)$, on the other hand, is called the double-well potential because $\varphi = \pm 1$ are its bi-stable critical points of the ordinary differential equation

$$\dot{\varphi} = \kappa(\varphi - \varphi^3).$$

The other critical point $\varphi = 0$ of $W = W(\varphi)$ is unstable, and, consequently, $\varphi = \varphi(\cdot,t)$ is rapidly separated into the regions $\{x \in \Omega \mid \varphi(x,t) = \pm 1\}$ with the interface $\{x \in \Omega \mid -1 < \varphi(x,t) < +1\}$ of $O(\xi)$-thickness, and then this interface moves slowly subject to their curvatures, that is, there are fast and slow dynamics.[35,77,153,266]

In fact, if $\varphi = \varphi(x)$ is regular, then this free energy is equal to

$$\mathscr{F}(\varphi) = \int_{-\infty}^{\infty} ds \int_{\{\varphi = s\}} \xi^2 |\nabla\varphi| + \frac{W(s)}{|\nabla\varphi|} \, dS$$

by the co-area formula. This value is estimated from below by

$$\mathscr{F}_*(\varphi) = 2\xi^2 \int_{-\infty}^{\infty} W(s) H^{n-1}(\{\varphi = s\})\, ds$$

with the equality if and only if

$$\frac{\partial \varphi}{\partial(\xi^{-1}\nu)} = W(s)^{1/2} \qquad \text{on } \{\varphi = s\},$$

where ν denotes the outer normal vector on $\{\varphi = s\}$ from $\{\varphi < s\}$ to $\{\varphi > s\}$ and H^{n-1} is the $(n-1)$-dimensional Hausdorff measure. Thus, in the equilibrium this φ is almost separated into $\varphi = \pm 1$, and the interface thickness is of $O(\xi)$.

To illustrate the relation between the mean curvature flow, we take $K = 1$ in (1.181), and assume

$$\varphi(x,t) \approx \phi\left(d(x,t)/\xi\right)$$

near the interface, denoted by Γ_t, where $d(x,t)$ is the signed-distance function. Then, a rough calculation implies

$$d_t \approx \Delta d,$$

where Δd and d_t are the mean curvature of Γ_t and the velocity of M_t, respectively. Thus, we otain the suggestion of the convergence of $\varphi(x,t)$ to the mean curvature flow as $\xi \downarrow 0$ of (1.181).
In the *level set approach*,[59,89,90] actually, this mean curvature flow is described by $\Gamma_t = \{u(\cdot,t)=0\}$ with $u=u(x,t)$ satisfying

$$u_t = |Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right) = \mathrm{trace}\left(I - \frac{Du\otimes Du}{|Du|^2}\right).$$

In spite of the formal singularity, this equation is well-posed in the sense of viscosity solutions,[68] and the above mentioned convergence is justified by the comparison principle under appropriate time scalings.[17,58,88,227] Also, we obtain efficient numerical schemes from this observation.[86,106,230,242]

Cahn-Hilliard Equation

Landau-Ginzburg's free energy induces also the Cahn-Hilliard equation[42]

$$\begin{aligned} &\varphi_t = -K\Delta(\xi^2\Delta\varphi - W'(\varphi)) && \text{in } \Omega\times(0,T) \\ &\frac{\partial}{\partial\nu}(\xi^2\Delta\varphi - W'(\varphi)) = 0 && \text{on } \partial\Omega\times(0,T) \end{aligned}$$

concerning the phase separation as a model (B) equation. Similarly to the above case of the Allen-Cahn equation, usually we impose

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\times(0,T)$$

furthermore, assuming that $\mathscr{F}(\varphi)$ is associated with all $\varphi\in H^1(\Omega)$ which means that

$$\begin{aligned} &\varphi_t = -K\Delta(\xi^2\Delta\varphi - W'(\varphi)) && \text{in } \Omega\times(0,T) \\ &\frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{1.183}$$

In this case, the stationary state φ is defined by

$$\begin{aligned} &-\xi^2\Delta\varphi = \varphi - \varphi^3 - \frac{1}{|\Omega|}\int_\Omega \varphi - \varphi^3\,dx \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \\ &\int_\Omega \varphi = \lambda, \end{aligned} \tag{1.184}$$

and its linearized stability means the positivity of the first eigenvalue of the self-adjoint operator in $L_0^2(\Omega)$

$$A = -\xi^2\Delta + 1 - 3\varphi^2$$

with the domain

$$D(A) = \left\{\psi\in H^2(\Omega) \mid \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \partial\Omega,\ \int_\Omega \psi = 0\right\}.$$

If $\varphi = \varphi_\xi$ attains the minimum of $\mathscr{F}_\xi$ defined by

$$\mathscr{F}_\xi(\varphi) = \begin{cases} \int_\Omega \frac{\xi^2}{2} |\nabla\varphi|^2 + W(\varphi)\, dx, \; \varphi \in H^1(\Omega), \; \int_\Omega \varphi = \lambda \\ +\infty, \quad \text{otherwise,} \end{cases}$$

then it is a solution to (1.184). Here, we obtain Γ-*convergence* of $\mathscr{F}_\xi$ to $\mathscr{F}_0$ in $L^1(\Omega)$ as $\xi \downarrow 0$, see,[205,289] where

$$\mathscr{F}_0(\varphi) = \begin{cases} \frac{2}{3}\mathrm{Per}_\Omega\{\varphi = -1\}, \; \varphi \in BV(\Omega), \; W(\varphi) = 0 \text{ a.e.}, \; \int_\Omega \varphi = \lambda \\ +\infty, \quad \text{otherwise,} \end{cases}$$

$BV(\Omega)$ denotes the space of functions of bounded variation,

$$BV(\Omega) = \left\{ v \in L^1(\Omega) \mid \int_\Omega |\nabla v| < +\infty \right\}$$
$$\int_\Omega |\nabla v| = \sup \left\{ \int_\Omega v\nabla \cdot g \mid g \in C_0^1(\Omega)^n, \;\; \|g\|_\infty \le 1 \right\},$$

and $\mathrm{Per}_\Omega(A) = \int_\Omega |\nabla \chi_A|$ is the perimeter of the measurable set A, see[125] which means that

$$v_\xi \to v \quad \text{as } \xi \downarrow 0 \text{ in } L^1(\Omega) \qquad \Rightarrow \qquad \liminf_{\xi\downarrow 0} \mathscr{F}_\xi(v_\xi) \ge \mathscr{F}_0(v)$$

and any $v \in L^1(\Omega)$ admits $\xi_k \downarrow 0$ and $\{v_k\}$ such that

$$v_k \to v \quad \text{in } L^1(\Omega)$$
$$\lim_{k\to\infty} \mathscr{F}_{\xi_k}(v_k) = \mathscr{F}_0(v).$$

In this case, any $\xi_k \downarrow 0$ admits $\{\xi_k'\} \subset \{\xi_k\}$ such that $\varphi_{\xi_k'}$ converges to a miminizer of $\mathscr{F}_0$ in $L^1(\Omega)$ from the genral theory.[70]
The limiting problem of (1.183) as $\xi \downarrow 0$ is described by the Mullins-Sekerka-Hele-Shaw equation,

$$\begin{aligned} &\Delta v = 0 && \text{in } \textstyle\bigcup_{t\in(0,T)} (\Omega \setminus \Gamma_t) \times \{t\} \\ &\frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\ &v = \gamma k_{\Gamma_t}, \;\; V = \frac{1}{2}\left[\frac{\partial v}{\partial \nu}\right]_-^+ && \text{on } \textstyle\bigcup_{t\in(0,T)} \Gamma_t \times \{t\}, \end{aligned}$$

where k_{Γ_t}, $\gamma > 0$, and V are the mean curvature of the interface Γ_t, a constant, and the normal velocity of Γ_t, respectively.[2,251,291]
The first factor of the micro-phase separation is the *spinodal decomposition* caused by the composition fluctuation of sine-like waves. It occurs to the blended material when the temperature becomes low. Its rough description is obtained by taking the equation on the whole space, and linearizing it around the constant solution $\overline{\varphi}$, that is

$$K^{-1} w_t = \Delta\left(-\xi^2 \Delta w + W''(\overline{\varphi}) w\right) \qquad \text{in } \mathbf{R}^n \times (0,T).$$

Thus putting

$$w(x,t) = \exp\left(\imath k \cdot x + K\sigma t\right),$$

we obtain

$$\sigma = |k|^2 \left(-W''(\overline{\varphi}) - \xi^2 |k|^2 \right),$$

where $\sigma > 0$ corresponds to this phenomenon of spinodal decomposition. This range is the case of $W''(\overline{\varphi}) < 0$, that is $-\frac{1}{\sqrt{3}} < \overline{\varphi} < \frac{1}{\sqrt{3}}$, and $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is called the *spinodal interval*, and then it holds that

$$|k|^2 = \frac{\beta \pm \sqrt{\beta^2 - 4\xi^2\sigma}}{2\xi^2},$$

where $\beta = -W''(\overline{\varphi})$. The value σ, on the other hand, attains the maximum if

$$|k|^2 = \frac{1}{2}\beta\xi^{-2}. \tag{1.185}$$

This formula suggests that unstable wave-like fluctuation with the length $\lambda = \frac{2\pi}{k} = \sqrt{\frac{8}{\beta}}\pi\xi$ in the one-dimensional case, and simultaneously, irregular patterns caused by the mixture of a variety of waves for the two-dimensional case.[41]

The second mechanism of phase separation is the *nucleation*, whereby several crystal seeds gather under the presense of the condensate of the solution, caused by super-saturation. It is associated with the intervals $I_- = (-1, -\frac{1}{\sqrt{3}})$ and $I_+ = (\frac{1}{\sqrt{3}}, 1)$ called the *metastable region*, and if the mean value of the order parameter is $\overline{\varphi} \in I_-$, then under the local but sufficiently strong perturbation induces a small region with $\varphi \in I_+$ surrounded by that with $\varphi \in I_-$, see.[3,19]

Ohta-Kawasaki's Free Energy

Ohta-Kawasaki's free energy,[231]

$$\mathscr{F}(\varphi) = \int_\Omega \frac{\xi^2}{2} |\nabla\varphi|^2 + W(\varphi) + \frac{\sigma}{2} \left| (-\Delta_N)^{-1/2}(\varphi - \overline{\varphi}) \right|^2 dx \tag{1.186}$$

is concerned with the micro-phase separation in diblock copolymers, where $\sigma > 0$ is a parameter proportional to the inverse length of the polymer chain and

$$\overline{\varphi} = \frac{1}{|\Omega|} \int_\Omega \varphi.$$

If the region is separated into $\varphi = \pm 1$ with the interface $O(\xi)$, then the third term of (1.186) is not negligible because the variation derived from $\overline{\varphi}$ becomes large. This term of Ohta-Kawasaki's free energy is essentially the same as that of Helmholtz' free energy defined by (1.16)

$$\mathscr{F}(u) = \frac{1}{2} \int_\Omega u(\log u - 1) - \frac{1}{2} \left\langle (-\Delta_{JL})^{-1} u, u \right\rangle.$$

It can be made small by the rapid oscillation that guarantees the convergence to $\overline{\varphi}$ of φ in $H^1(\Omega)'$ which suggests a number of local minima of this functional.

The free energy (1.186) induces the Nishiura-Ohnishi equation[228]

$$\begin{aligned} &\alpha\varphi_t = -\Delta\left(\xi^2\Delta\varphi - W'(\varphi)\right) - \sigma(\varphi - \overline{\varphi}) && \text{in } \Omega\times(0,T) \\ &\frac{\partial}{\partial\nu}\left\{\xi^2\Delta\varphi - W'(\varphi)\right\} = 0 && \text{on } \partial\Omega\times(0,T) \end{aligned}$$

as a model (B) equation, where $\alpha = K^{-1}$. Similarly to the former cases, we impose

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\times(0,T)$$

using all $\varphi \in H^1(\Omega)$ to derive $\delta\mathscr{F}(\varphi)$, and then it holds that

$$\begin{aligned} &\varphi_t = -\Delta\left(\xi^2\Delta\varphi - W'(\varphi)\right) - \sigma(\varphi - \overline{\varphi}) && \text{in } \Omega\times(0,T) \\ &\frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned}$$

The stationary state is described by

$$\begin{aligned} &-\xi^2\Delta\varphi = \varphi - \varphi^3 - \frac{1}{|\Omega|}\int_\Omega \varphi - \varphi^3\, dx + \sigma(-\Delta_{JL})^{-1}\varphi \quad \text{in } \Omega \\ &\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega, \qquad \int_\Omega \varphi = \lambda. \end{aligned}$$

Then, the linearized stablity of this stationary state means the positivity of the first eigenvalue of the self-adjoint operator A in $L^2_0(\Omega)$ defined by

$$A\psi = -\xi^2\Delta\psi - \psi + 3\varphi^2\psi + \sigma(-\Delta_{JL})^{-1}\psi$$

with the domain

$$D(A) = \left\{\psi \in H^2(\Omega) \mid \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \partial\Omega, \ \int_\Omega \psi = 0\right\}.$$

We have also Γ-convergence of the functional $\mathscr{F} = \mathscr{F}_{\xi,\sigma}$ of (1.186) under several scalings of (ξ,σ), see.[243,244]

1.3.2. Gradient and Skew-Gradient Systems

There are several systems derived from Lagrangian and skew-Lagrangian without duality. In this case, the stationary state is not reduced to the single equation.

In the gradient system, the Lagrangian $L = L(u,v)$ acts as a Lyapunov function. If the combination of model (A) - model (B) equations

$$\begin{aligned} u_t &= -L_u \\ \tau v_t &= -L_v, \end{aligned}$$

for example, is adopted, then it holds that

$$\begin{aligned} \frac{d}{dt}L(u,v) &= -\int_\Omega L_u(u,v)^2 + \tau^{-1}L_v(u,v)^2\, dx \\ &\leq 0, \end{aligned}$$

where $\tau > 0$ is a constant. The stationary state $(\overline{u}, \overline{v})$ is defined by

$$L_u(\overline{u}, \overline{v}) = L_v(\overline{u}, \overline{v}) = 0,$$

and its linearized stability is described by the positivity of

$$A = \begin{pmatrix} L_{uu}(\overline{u}, \overline{v}) & L_{uv}(\overline{u}, \overline{v}) \\ L_{vu}(\overline{u}, \overline{v}) & L_{vv}(\overline{u}, \overline{v}) \end{pmatrix}$$

provided that $L = L(u, v)$ is C^2 because the linearized equation is given by

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} + JA\begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{pmatrix}. \tag{1.187}$$

This operator A is the same as the Hessian of L and the linearly stable stationary solution derived from this Lagrangian is dynamically stable. If $L_{uv}(\overline{u}, \overline{v}) = 0$, then this linearized stability is equivalent to the component-wise positivities of the operators $L_{uu}(\overline{u}, \overline{v})$ and $L_{vv}(\overline{u}, \overline{v})$,

$$\begin{aligned} L_{uu}(\overline{u}, \overline{v}) &> 0 \\ L_{vv}(\overline{u}, \overline{v}) &> 0. \end{aligned} \tag{1.188}$$

For the Toland duality with C^2 Lagrangian

$$L(u, v) = F^*(u) + G(v) - \langle v, u \rangle,$$

this stability holds if $(F^*)''(\overline{u}) > 1$ and $G''(\overline{v}) > 1$ as quadratic forms.
In the skew-gradient system using the skew-Lagrangian $L = L(u, v)$, that is

$$\begin{aligned} u_t &= -L_u \\ \tau v_t &= L_v, \end{aligned}$$

the stationary state is similarly defined by

$$L_u(\overline{u}, \overline{v}) = L_v(\overline{u}, \overline{v}) = 0,$$

while the linearized equation is indicated by (1.187) for

$$A = \begin{pmatrix} L_{uu}(\overline{u}, \overline{v}) & L_{uv}(\overline{u}, \overline{v}) \\ -L_{uv}(\overline{u}, \overline{v}) & L_{vv}(\overline{u}, \overline{v}) \end{pmatrix}.$$

In this case, the linearized stablity of $(\overline{u}, \overline{v})$ means that any eigenvalues of A is in the right-half space, or, equivalently,

$$\text{Re } (JAw, w)_J = \text{Re } (Aw, w) > 0 \qquad \text{for all } w = \begin{pmatrix} u \\ v \end{pmatrix} \neq 0 \tag{1.189}$$

using the complex-valued trial functions $\{u, v\}$ and the L^2-inner product

$$(w_1, w_2) = \int_\Omega u_1 u_2^* + v_1 v_2^* \, dx$$

for

$$w_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad i = 1,2,$$

where z^* denotes the complex-conjugate of $z \in \mathbf{C}$ and

$$(w_1, w_2)_J = (J^{-1/2}w_1, J^{-1/2}w_2).$$

Since $(\overline{u}, \overline{v})$ is real-valued, it holds that

$$\begin{aligned} \text{Re } (Aw, w) &= \langle u, L_{uu}(\overline{u}, \overline{v})u \rangle + \langle v, L_{vv}(\overline{u}, \overline{v})v \rangle \\ \langle w_1, w_2 \rangle &= \int_\Omega u_1^* u_2 + v_1^* v_2 \, dx \end{aligned}$$

and, therefore, condition (1.189) is equivalent to (1.188) and such a stationary state is dynamically stable if $L = L(u,v)$ is C^2, see.[346,347]

Eguchi-Oki-Matsumura Equation

An example of the gradient system without duality is the Eguchi-Oki-Matsumura equation concerning phase separation of alloys.[82] It is a combination of model (B) and model (A) equations, using the Lagrangian

$$L(u,v) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{\xi^2}{2} |\nabla v|^2 + f(u,v) \, dx,$$

where

$$f(u,v) = \frac{a}{2}u^2 - \frac{b}{2}v^2 + \frac{b'}{4}v^4 + \frac{g}{2}u^2 v^2$$

with $\tau, a, b, b', g > 0$ constants, stands for the bulk-energy and u and v are the concentration of the main component and the order parameter, respectively. Thus, we obtain

$$\begin{aligned} &\tau u_t = \nabla \cdot \nabla L_u(u,v) \\ &v_t = -L_v(u,v) && \text{in } \Omega \times (0,T) \\ &\frac{\partial}{\partial \nu} L_u(u,v) = 0 && \text{on } \partial\Omega \times (0,T), \end{aligned}$$

and hence

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u = 0 \\ &\frac{d}{dt} L(u,v) = -\int_\Omega \tau^{-1} |\nabla L(u,v)|^2 + v_t^2 \, dx \le 0, \end{aligned}$$

where $\tau > 0$ is a constant.
Using all $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$ in deriving L_u and L_v, we obtain

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

as a natural boudary condition. Then the stationary state is described by

$$-\Delta u + au + guv^2 = \text{constant}, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega u = \lambda$$
$$-\xi^2 \Delta v - bv + b'v^3 + gu^2 v = 0, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0.$$

This system is difficult to reduce to single equations on u or v. There arise, however, multiple stationary solutions.[133]

Gierer-Meinhardt Equation

Yanagida[346,347] formulated the Gierer-Meinhardt equation and the HitzHugh-Nagmo equation, see §1.3.4, as skew-gradient systems. The former is concerned with the morphogenesis[118] and is a combination of model (A) equations

$$ra_t = -L_a$$
$$q\tau h_t = L_h,$$

where $r, q, \tau > 0$ are constants. It is derived from the skew-Lagrangian

$$L(a,h) = \int_\Omega \frac{r\varepsilon}{2} |\nabla a|^2 - \frac{qD}{2} |\nabla h|^2 - H(a,h)\, dx$$

with

$$H(a,h) = -\frac{r}{2}a^2 + r\sigma a + a^{p+1}h^q + \frac{q}{2}h^2$$

in the case of

$$p+1 = r, \qquad q+1 = s, \tag{1.190}$$

where $\varepsilon, D, \sigma > 0$ are constants. If all $a \in H^1(\Omega)$ and $h \in H^1(\Omega)$ are taken to derive L_a and L_h, then we obtain

$$\begin{aligned} a_t &= \varepsilon^2 \Delta a - a + \frac{a^p}{h^q} + \sigma \\ \tau h_t &= D\Delta h - h + \frac{a^r}{h^s} \qquad \text{in } \Omega \times (0,T) \\ \frac{\partial a}{\partial \nu} &= \frac{\partial h}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{1.191}$$

The shadow system is obtained by making $D \to +\infty$ in the second equation. More precisely, in this case $h = h(t)$ is independent of x, and then it holds that

$$\begin{aligned} a_t &= \varepsilon^2 \Delta a - a + a^p/h^q \\ \tau h_t &= -h + \frac{1}{h^s} \cdot \frac{1}{|\Omega|} \int_\Omega a^r \qquad \text{in } \Omega \times (0,T) \\ \frac{\partial a}{\partial \nu} &= \frac{\partial h}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{1.192}$$

by operating $\frac{1}{|\Omega|}\int_\Omega \cdot$, where $\sigma = 0$ is assumed for simplicity. Then its stationary state is defined by

$$\varepsilon^2 \Delta a - a + \frac{a^p}{h^q} = 0$$
$$h^{s+1} = \frac{1}{|\Omega|}\int_\Omega a^r \qquad \text{in } \Omega$$
$$\frac{\partial a}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

see.[168,226] In the case of $r = p+1$ there is a variational functional

$$J(v) = \frac{1}{2}\int_\Omega \varepsilon^2 |\nabla a|^2 + a^2 \, dx - \frac{1}{(1-\gamma)r}\left(\int_\Omega a^r\right)^{1-\gamma}$$

defined for $v \in H^1(\Omega)$, where $\gamma = \frac{r}{s+1}$. General case other than (1.190) is not formulated by the skew-gradient system.
We have spiky stationary solutions, slow dynamics of spikes, and Hopf bifurcation to (1.191), see.[72,334,335] The skew-gradient system can thus be involved by the top-down self-organization, while the stable stationary solutions take a role of the bottom-up self-organization.

1.3.3. Skew-Gradient Systems with Duality

If the skew-Lagrangian is defined by the free energy using Kuhn-Tucker duality, then the stationary states split into the particle and the field components provided with the structure of dual variation. This property is a special case with the convexity of the functional, and the global dynamics near the stationary state is not so complicated.
Let X be a Banach space over $\mathbf{R}$, and $F, G : X \to (-\infty, +\infty]$ be proper, convex, lower semi-continuous functionals. The Lagrangian in the Toland duality is defined by

$$L(x,p) = G(x) + F^*(p) - \langle x, p\rangle$$

for $(x,p) \in X \times X^*$, see §1.2.3, where X^* denotes the dual space and F^* is the Legendre transformation of F:

$$F^*(p) = \sup_{x\in X}\left\{\langle x,p\rangle - F(x)\right\}.$$

It is associated with the ”free energy“

$$J^*(p) = \begin{cases} F^*(p) - G^*(p), & p \in D(F^*) \\ +\infty, & \text{otherwise} \end{cases}$$

and the ”field functional“

$$J(x) = \begin{cases} G(x) - F(x), & x \in D(G) \\ +\infty, & \text{otherwise} \end{cases}$$

through

$$J^*(p) = \inf_{x\in X} L(x,p)$$
$$J(x) = \inf_{p\in X^*} L(x,p).$$

Then we obtain the unfolding-minimality (1.145)-(1.147). Furthermore, $\overline{p}$ and $\overline{x}$ are linearly stable critial points of J^* and J, respectively, if and only if $(\overline{x},\overline{p})$ is a linearly stable critical point of L, and the former two conditions are equivalent each other by $\overline{p} \in \partial G(\overline{x}) \cap \partial F(\overline{x})$ and $\overline{x} \in \partial G^*(\overline{p}) \cap \partial F^*(\overline{p})$.
Skew-Lagrangian with duality, on the other hand, is defined by

$$L(x,p) = \begin{cases} F^*(p) - G(x) + \langle x,p\rangle, & p \in D(F^*),\ x \in X \\ +\infty, & \text{otherwise} \end{cases}$$

Then, putting

$$J(x) = G(x) + F(-x)$$
$$J^*(p) = F^*(p) + G^*(p),$$

we obtain a similar structure. In other words, we say that $(\overline{x},\overline{p}) \in X \times X^*$ is a critical point of L if

$$0 \in L_x(\overline{x},\overline{p})$$
$$0 \in L_p(\overline{x},\overline{p}),$$

which is equivalent for $\overline{x}$ and $\overline{p}$ to be critical points of J and J^*, respectively.

$$\overline{p} \in \partial F(-\overline{x}) \cap \partial G(\overline{x})$$
$$\overline{x} \in -\partial F^*(\overline{p}) \cap \partial G^*(\overline{p}). \tag{1.193}$$

These two relations of (1.193) are equivalent each other, and if one of them is satisfied then it holds that

$$L(\overline{x},\overline{p}) = J^*(\overline{p}) = -J(\overline{x}).$$

We have the unfolding

$$L|_{x\in\partial G^*(p)} = J^*$$
$$L|_{p\in\partial F(-x)} = -J$$

and the "minimality" in the sense of

$$J^*(p) \le L(x,p) \le -J(x)$$

for any $(x,p) \in X \times X^*$.
A critical point $(\overline{x},\overline{p})$ is called a *saddle* if

$$L(x,\overline{p}) \le L(\overline{x},\overline{p}) \le L(\overline{x},p) \tag{1.194}$$

for any (x,p) in a neighborhood of $(\overline{x},\overline{p})$. It is linearly stable if there is $\varepsilon_0 > 0$ such that any $\varepsilon \in (0,\varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{lcl} \|x-\overline{x}\|_X < \varepsilon_0 & & \\ \|p-\overline{p}\|_{X^*} < \varepsilon_0 & \Rightarrow & \|x-\overline{x}\|_X < \varepsilon \\ L(\overline{x},\overline{p}) - L(x,p) < \delta & & \|p-\overline{p}\|_{X^*} < \varepsilon. \\ L(x,p) - L(\overline{x},\overline{p}) < \delta & & \end{array}$$

These conditions of stability split component-wisely, and $(\overline{x},\overline{p})$ is a linearly stable saddle of L if $\overline{x}$ and $\overline{p}$ are linearly stable critical points of J and J^*, respectively, and they are equivalent each other. More precisely, these $\overline{u}$ and $\overline{v}$ are linearly stable if there exists $\varepsilon_0 > 0$ such any $\varepsilon \in (0,\varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{lcl} \|x-\overline{x}\|_X < \varepsilon_0 & \Rightarrow & \|x-\overline{x}\|_X < \varepsilon \\ J(x) - J(\overline{x}) < \delta & & \end{array}$$

and

$$\begin{array}{lcl} \|p-\overline{p}\|_{X^*} < \varepsilon_0 & \Rightarrow & \|p-\overline{p}\|_{X^*} < \varepsilon, \\ J^*(p) - J^*(\overline{p}) < \delta & & \end{array}$$

respectively.

The skew-gradient system

$$\begin{aligned} \dot{p} &\in -L_p(x,p) \\ \tau\dot{x} &\in L_x(x,p) \end{aligned} \tag{1.195}$$

derived from the skew-Lagrangian $L = L(x,p)$ with duality, defined above, has similar properties to (1.165). Thus in the presense of the Hilbert space Y satisfying (1.166), the solution $(x(t),p(t))$ in (1.167) satisfies

$$\begin{aligned} &\limsup_{h\downarrow 0} \frac{1}{h}\left\{F^*(p(t)) - F^*(p(t-h))\right\} \leq -\langle x,\dot{p}\rangle - \|\dot{p}(t)\|_Y^2 \\ &\limsup_{h\downarrow 0} \frac{1}{h}\left\{G(x(t)) - G(x(t-h))\right\} \leq \langle \dot{x},p\rangle - \tau\|\dot{x}(t)\|_Y^2, \end{aligned}$$

and, therefore,

$$\begin{aligned} &\limsup_{h\downarrow 0} \frac{1}{h}\left\{L(x(t),p(t)) - L(x(t),p(t-h))\right\} \leq -\|\dot{p}(t)\|_Y^2 \\ &\limsup_{h\downarrow 0} \frac{1}{h}\left\{-L(x(t),p(t)) + L(x(t-h),p(t))\right\} \leq -\tau\|\dot{x}(t)\|_Y^2. \end{aligned}$$

Thus, if $(\overline{x},\overline{p}) \in Y \times Y$ is a linearly stable saddle point of L and (1.195) is locally well-posed in its neighborhood in the sense of (1.167), then this skew-gradient system is well-posed globally in time there, $(\overline{x},\overline{p})$ is dynamically stable, and it holds that (1.164).

1.3.4. Semi-Unfolding-Minimality

As is described in the previous paragraph, if L is a skew-Lagrangian with duality, then both J^* and J are convex, and hence any critical point of L is a saddle. Several other systems, however, are provided only with the semi-duality, and admit multiple stationary solutions. In this paragraph, we describe the Lagrangian and the skew-Lagrangian combined with convex and non-convex functionals, which are found in several phase field models provided with the double-well potential, see §1.5.

Semi-Lagrangian

First, taking a Banach space X over $\mathbf{R}$, we define the Lagrangian with semi-duality by

$$L(x,p) = \begin{cases} F^*(p) + G(x) - \langle x, p \rangle, & x \in D(G) \\ +\infty, & \text{otherwise}, \end{cases}$$

where

$$G : X \to (-\infty, +\infty]$$

is proper, convex, lower semi-continuous, while

$$F^* : X^* \to [-\infty, +\infty]$$

is *arbitrary*. The semi-critical point $(\overline{x}, \overline{p})$ is defined by

$$L_p(\overline{x}, \overline{p}) = 0$$

or, equivalently,

$$\overline{p} \in \partial G(\overline{x}). \tag{1.196}$$

Here, $p \in \partial G(x)$ is equivalent to $x \in \partial G^*(p)$, and then it holds that

$$G^*(p) - \langle x, p \rangle = -G(x).$$

Thus, we obtain the semi-unfolding-minimality

$$\inf_x L = L|_{x \in \partial G^*(p)} = J^*,$$

where

$$J^*(p) = \begin{cases} F^*(p) - G^*(p), & p \in D(F^*) \\ +\infty, & \text{otherwise}. \end{cases} \tag{1.197}$$

Given a critical point of J^*, denoted by $\overline{p}$, we define its linearized stability similarly, that is $\overline{p} \in X^*$ of

$$\delta J^*(\overline{p}) = 0$$

is linearly stable if there is $\varepsilon_0 > 0$ such that any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{matrix} \|p - \overline{p}\|_{X^*} < \varepsilon_0 \\ J^*(p) - J^*(\overline{p}) < \delta \end{matrix} \quad \Rightarrow \quad \|p - \overline{p}\|_{X^*} < \varepsilon,$$

and such $\overline{p}$ is dynamically stable. More precisely, if $\overline{p}$ is a linearly stable critical point of J^*, then any $\varepsilon > 0$ admits $\delta > 0$ such that if $(x(t), p(t)) \in X \times X^*$ $(0 \le t < T)$ satisfies

$$\begin{aligned} & t \in [0,T) \mapsto p(t) \in X^* && \text{continuous} \\ & t \in [0,T) \mapsto L(x(t), p(t)) \in \mathbf{R} && \text{non-increasing} \\ & \|p(0) - \overline{p}\|_{X^*} < \delta, \end{aligned}$$

then it holds that

$$\sup_{t\in[0,T)} \|p(t) - \overline{p}\|_{X^*} < \varepsilon.$$

We have

$$\begin{aligned} L(x,p) &= J^*(p) + \hat{L}(x,p) \\ \hat{L}(x,p) &= G^*(p) + G(x) - \langle x, p \rangle \ge 0 \end{aligned}$$

and $\hat{L}(x,p) = 0$ holds if and only if $x \in \partial G^*(p)$, or, equivalently, $p \in \partial G(x)$. Given a critical point $\overline{p} \in X^*$ of J^*, we take

$$\overline{x} \in \partial G^*(\overline{p}),$$

and now define its *hyper-linear stability*. Thus $\overline{x} \in \partial G^*(\overline{p})$ is hyper-linearly stable if there is $\varepsilon_0 > 0$ such that any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{l} \|x - \overline{x}\|_X < \varepsilon_0 \\ \hat{L}(x,p) - \hat{L}(\overline{x},\overline{p}) < \delta \\ \|p - \overline{p}\|_{X^*} < \delta \end{array} \quad \Rightarrow \quad \|x - \overline{x}\|_X < \varepsilon. \tag{1.198}$$

Consequently, if $\overline{p}$ is a linearly stable critical point of J^* and $\overline{x} \in \partial G^*(\overline{p})$ is hyper-linearly stable, then any $\varepsilon > 0$ admits $\delta > 0$ such that

$$\begin{aligned} & t \in [0,T) \mapsto (x(t), p(t)) \in X \times X^* && \text{continuous} \\ & t \in [0,T) \mapsto L(x(t), p(t)) \in \mathbf{R} && \text{non-increasing} \\ & \|x(0) - \overline{x}\|_X + \|p(0) - \overline{p}\|_{X^*} < \delta \\ & \Rightarrow \\ & \sup_{t\in[0,T)} \{\|x(t) - \overline{x}\|_X + \|p(t) - \overline{p}\|_{X^*}\} < \varepsilon. \end{aligned}$$

Semi-Skew-Lagrangian

Skew-Lagrangian with semi-duality is defined by

$$L(x,p) = \begin{cases} F^*(p) - G(x) + \langle x, p \rangle, & p \in D(F^*) \\ +\infty, & \text{otherwise}, \end{cases} \tag{1.199}$$

where $G : X \to (-\infty, +\infty]$ is proper, convex, lower semi-continuos, while $F^* : X^* \to [-\infty, +\infty]$ is *arbitrary*. A semi-critical point $(\overline{x}, \overline{p})$ of L is defined by (1.196), and there is semi-unfolding-minimality,

$$\sup_x L = L|_{x\in\partial G^*(p)} = J^* \tag{1.200}$$

for the free energy defined by

$$J^*(p) = F^*(p) + G^*(p). \tag{1.201}$$

Linearized stability of a critical point $\overline{p} \in X^*$ of J^* is defined by (1.198). Here, we note that the multiple existence of such $\overline{p}$ can occur because F^* may not be convex. A stronger concept of the *robust stability* of $\overline{p}$ indicates the existence of $\varepsilon_0 > 0$ such that any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{l} \|x - \overline{x}\|_X + \|p - \overline{p}\|_{X^*} < \varepsilon_0 \\ L(x,p) - J^*(\overline{p}) < \delta \end{array} \qquad \Rightarrow \qquad \|p - \overline{p}\|_{X^*} < \varepsilon.$$

Although there is neither the Lyapunov function nor the anti-Lyapunov function in the skew-gradient system, such $\overline{p}$ is stable as far as $x(t)$ stays nearby $\overline{x}$, where $(\overline{x}, \overline{p})$ is a semi-critical point of L so that $\overline{x} \in \partial G^*(\overline{p})$. Thus, the driving force of leaving such $(\overline{x}, \overline{p})$ lies in the x component. We emphasize that this "semi-stability" is different from the (full)-stability of $(\overline{x}, \overline{p})$ defined in §1.3.3 for C^2 skew-Lagrangian.
Since

$$\begin{aligned} L(x,p) &= J^*(p) - \hat{L}(x,p) \\ \hat{L}(x,p) &= G^*(p) + G(x) - \langle x, p \rangle, \end{aligned}$$

the hyper-linear stability of $\overline{x} \in \partial G^*(\overline{p})$ is defined similarly, where $\overline{p}$ is a critical point of J^* defined by (1.201). It holds that

$$L(x,p) - J^*(\overline{p}) = J^*(p) - J^*(\overline{p}) - \hat{L}(x,p) < \delta$$

and, therefore, if $\overline{p} \in X^*$ is a linearly stable critical point of J^* defined by (1.201) and $\overline{x} \in \partial G^*(\overline{p})$ is hyper-linearly stable, then $\overline{p}$ is robust stable. The skew-gradient system derived from this semi-skew-Lagrangian, on the other hand, has the properties

$$\begin{array}{ll} t \in [0,T) \mapsto (x(t), p(t)) \in X \times X^* & \text{continuous} \\ t \in [0,T) \mapsto L(x, p(t)) \in \mathbf{R} & \text{non-increasing if } \|x - \overline{x}\|_X < \varepsilon_0 \\ t \in [0,T) \mapsto L(x(t), p) \in \mathbf{R} & \text{non-decreasing if } \|p - \overline{p}\|_{X^*} < \varepsilon_0 \end{array}$$

with $\varepsilon_0 > 0$, and, therefore, if $\overline{x} \in G^*(\overline{p})$ is hyper-linearly stable, then $(x(t), p(t))$ stays nearby $(\overline{x}, \overline{p})$ as far as $\|p(t) - \overline{p}\|_{X^*} < \varepsilon_0$ is kept.
Thus if $\overline{p} \in X^*$ is a linearly stable critical point of J^* defined by (1.201) and $\overline{x} \in \partial G^*(\overline{p})$ is hyper-linearly stable, then $(\overline{x}, \overline{p})$ is dynamically stable, and, in this way, the stationary solution and its full-stability split into each component in the skew-gradient system with semi-duality. If $L(x,p)$ is C^2, then this condition is reduced to (1.188).

FitzHugh-Nagumo Equation

The FitzHugh-Nagumo equation concerning nerve impluse[98,214] is a combination of model (A) equations

$$\begin{aligned} u_t &= -L_u(u,v) \\ \tau v_t &= L_v(u,v), \end{aligned}$$

where $\tau > 0$ is a constant. It is derived from the skew-Lagrangian

$$L(v,u) = \int_\Omega \frac{\xi^2}{2}|\nabla u|^2 + W(u) - \frac{\sigma}{2}|\nabla v|^2 + uv\, dx$$

defined for

$$u \in H^1(\Omega), \quad v \in H^1(\Omega), \quad \int_\Omega v = 0,$$

where $\xi, \sigma > 0$ are constants and

$$W(u) = \frac{1}{4}\left(u^2 - 1\right)^2.$$

Thus, we obtain

$$\begin{aligned}
& u_t = \xi^2 \Delta u - W'(u) - v \\
& \tau v_t = \sigma \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u && \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\
& \int_\Omega v = 0, && 0 < t < T.
\end{aligned} \tag{1.202}$$

The above skew-Lagrangian is provided with the semi-duality, and (1.199) holds with

$$\begin{aligned}
& F^*(u) = \int_\Omega \frac{\xi^2}{2}|\nabla u|^2 + W(u)\, dx, && u \in X \\
& G(v) = \frac{\sigma}{2}\|\nabla v\|_2^2, && v \in X, \ \int_\Omega v = 0,
\end{aligned}$$

where $X = H^1(\Omega)$. Then the free energy is defined by (1.197), that is

$$J^*(u) = \int_\Omega \frac{\xi^2}{2}|\nabla u|^2 + W(u)\, dx + \frac{\sigma}{2}\left\langle (-\Delta_{JL})^{-1}u, u \right\rangle,$$

which is nothing but Ohta-Kawasaki's free energy $\mathscr{F}(u)$ defined by (1.186).
The stationary state of (1.202) is, in particular, given by

$$\delta \mathscr{F}(u) = 0$$

and $v = (-\Delta_{JL})^{-1}u$. The former is equivalent to

$$\begin{aligned}
& -\xi^2 \Delta u = u - u^3 + \sigma(-\Delta_{JL})^{-1}u && \text{in } \Omega \\
& \frac{\partial u}{\partial \nu} = 0 && \text{on } \partial\Omega,
\end{aligned}$$

and we can confirm the semi-unfolding-minimality

$$\sup_{v,\ \int_\Omega v = 0} L = L|_{v=(-\Delta_{JL})^{-1}u} = \mathscr{F}. \tag{1.203}$$

Since the FitzHugh-Nagumo equation (1.202) is skew-gradient, the linearized stability of the stationary state (u,v) is reduced to the positivities of $L_{uu}(u,v)$ and $L_{vv}(u,v)$. The latter positivity is obvious because $L_{vv}(u,v)$ is nothing but $-\Delta$ with the domain

$$\left\{ \psi \in H^2(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \ \int_\Omega \psi = 0 \right\},$$

and, therefore, its linearized stability is described by the positivity of the first eigenvalue of

$$A_{FHN} = -\xi^2 \Delta - 1 + 3u^2$$

with the domain

$$D(A_{FHN}) = \left\{ \psi \in H^2(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}.$$

The first eigenvalue of this operator is denoted by $\mu_{FHN}(u)$.
The stationary state u is the same as that of the model (A) equation derived from Ohta-Kawasaki's free energy. In the latter case, its linearized stability is indicated by the positivity of the first eigenvalue of A_{OK} defined by

$$A_{OK}\psi = -\xi^2 \Delta\psi - \psi + 3u^2\psi + \sigma(-\Delta_{JL})^{-1}\psi$$

with the domain

$$D(A_{OK}) = \left\{ \psi \in H^2(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}.$$

Thus if its first eigenvalue is denoted by $\mu_{OK}(u)$, then we obtain

$$\mu_{OK}(u) > \mu_{FHN}(u).$$

This relation, combined with the semi-unfolding (1.203), indicates that the instability around a stationary state (u,v) of the FitzHugh-Nagumo equation satisfying

$$\mu_{OK}(u) > 0 > \mu_{HFN}(u)$$

occurs to v at the begining, casting the driving force to the self-organization.

1.3.5. Kuhn-Tucker Duality

Skew-Lagrangian with duality is regarded as a special case of the Kuhn-Tucker duality,[83,106,321] and this paragraph is devoted to the description of the latter.
Let X be a Banch space over **R** and regard

$$L = L(x,p) : X \times X^* \to [-\infty, +\infty]$$

as a skew-Lagrangian. In the context of game theory, L stands for the price which X and X^* wish to increase and decrease, respectively. Then $\alpha = \sup_x \inf_p L(x,p)$ indicates the best price of X when X^* takes his own best strategy against X. Similarly, $\beta = \inf_p \sup_x L(x,p)$ is the best price of X^* when X takes his own best strategy against X^*. It is obvious that $\alpha \leq \beta$, but a sufficient condition to $\alpha = \beta$ is the exsitence of the saddle $(\overline{x}, \overline{p}) \in X \times X^*$ defined by

(1.194). Then von Neumann's theorem guarantees the existence of such a saddle, provided that

$$x \in X \mapsto L(x,p)$$

is proper, concave, upper semi-continuous, and coercive from above for any $p \in X^*$, and simultaneously,

$$p \in X^* \mapsto L(x,p)$$

is proper, convex, lower semi-continuous, and coercive from below for any $x \in X$, see[83,303] for instance.
If the cost function

$$\varphi = \varphi(x,y) : X \times X \to [-\infty, +\infty]$$

is given, then the principal and the dual problems are defined by

$$\begin{array}{ll} (P) & \inf_{x \in X} \varphi(x,0) \\ (P^*) & \sup_{q \in X^*} -\varphi^*(0,q), \end{array}$$

where

$$\varphi^*(p,q) = \sup_{(x,y) \in X \times X} \{\langle x,p \rangle + \langle y,q \rangle - \varphi(x,y)\}$$

denotes the Legendre transformation of $\varphi(x,y)$. Then the duality theorem of Kuhn-Tucker guarantees the same value of (P) and (P^*), provided that

$$y \in X \mapsto \Phi(y) = \inf_x \varphi(x,y)$$

is proper, convex, lower semi-continuous.
In fact, we have

$$\begin{aligned} \varphi^*(0,q) &= \sup_{x,y \in X} \{\langle y,q \rangle - \varphi(x,y)\} \\ &= \sup_{y \in X} \{\langle y,q \rangle - \Phi(y)\} = \Phi^*(q) \end{aligned}$$

and hence

$$\begin{aligned} \sup_{q \in X^*} -\varphi^*(0,q) &= \sup_{q \in X^*} -\Phi^*(q) = \Phi^{**}(0) \\ &= \Phi(0) = \inf_{x \in X} \varphi(x,0) \end{aligned}$$

by Fenchel-Moreau's duality.
Given the cost function $\varphi = \varphi(x,y)$, we define the skew-Lagrangian by

$$L(x,q) = \sup_{y \in X} \{\langle y,q \rangle - \varphi(x,y)\}, \tag{1.204}$$

that is the Legendre transformation of

$$y \in X \mapsto \varphi_x(y) = \varphi(x,y).$$

If this mapping is proper, convex, lower semi-continuous, then we obtain

$$\varphi(x,y) = \varphi_x^{**}(y) = \sup_{q\in X^*} \{\langle y,q\rangle - L(x,q)\},$$

similarly, and, therefore, the principal problem is described by

$$\inf_{x\in X} \varphi(x,0) = -\sup_{x\in X} \inf_{q\in X^*} L(x,q).$$

We have, on the other hand,

$$\varphi^*(p,q) = \sup_{x\in X} \{\langle x,p\rangle + L(x,q)\} \tag{1.205}$$

and hence

$$\sup_{q\in X^*} -\varphi^*(0,q) = -\inf_{q\in X^*} \sup_{x\in X} L(x,q).$$

Using the duality theory mentioned above, thus, we obtain Kuhn-Tucker's saddle point theorem.

Theorem 1.25. *If $\Phi = \Phi(y)$ and $\varphi_x = \varphi_x(y)$ defined above are proper, convex, lower semi-continuous, and if the principal and the dual problems have the solutions $\overline{x}$ and $\overline{p}$, respectively, then $(\overline{x},\overline{p})$ is a saddle of L.*

Skew-Lagrangian with Duality

If $p \mapsto L(x,p)$ is proper, convex, lower semi-continuous for each x, coversely, then the cost function $\varphi = \varphi(x,y)$ defined by

$$\varphi(x,y) = \sup_{p} \{\langle y,p\rangle - L(x,p)\}$$

satisfies (1.204), and then equality (1.205) holds true. In this case, the principal and the dual problems are formulated by

$$\begin{aligned} &(P) \quad \inf_x \varphi(x,0) = -\sup_x \inf_q L(x,q) \\ &(P^*) \quad \sup_q -\varphi^*(0,q) = -\inf_q \sup_x L(x,q), \end{aligned}$$

and, in particular, these problems have the same value if $L = L(x,p)$ has a saddle, (1.194). The skew-Lagrangian with duality in §1.3.3 is described by

$$L(x,p) = F^*(p) - G(x) + \langle x,p\rangle$$

with $F,G : X \to (-\infty,+\infty]$, proper, convex, lower semi-continuous. In this case, the cost function $\varphi(x,y)$ and its Legendre transformation $\varphi^*(p,q)$ are defined by

$$\begin{aligned} \varphi(x,y) &= \sup_p \{\langle y,p\rangle - L(x,p)\} = F(y-x) + G(x) \\ \varphi^*(p,q) &= \sup_{x,y} \{\langle x,p\rangle + \langle y,q\rangle - \varphi(p,q)\} = F^*(q) + G^*(p+q), \end{aligned}$$

and, therefore, the principal and the dual problems are described by

$$(P)\quad \inf_x \varphi(x,0) = \inf_x \{G(x)+F(-x)\} = \inf_x J(x)$$
$$(P^*)\quad \sup_q -\varphi^*(0,q) = \inf_q \{F^*(q)+G^*(q)\} = \inf_q J^*(q).$$

Concerning the existence of the saddle, we can apply the theorem of von Neumann, and then obtain the following theorem.

Theorem 1.26. *If X is a Banach space over $\mathbf{R}$, and $F,G : X \to (-\infty,+\infty]$ are proper, convex, lower semi-continuous, then the skew-Lagrangian*

$$L(x,p) = F^*(p) - G(x) + \langle x,p\rangle$$

has a saddle if

$$x \mapsto G(x) - \langle x,\overline{p}\rangle$$
$$p \mapsto F^*(p) - \langle \overline{x},p\rangle$$

are coercive for each $(\overline{x},\overline{p})$. In this case, the associated variational problems

$$\inf_x J(x)$$
$$\inf_p J^*(p)$$

have the same value, where

$$J(x) = G(x) + F(-x)$$
$$J^*(p) = F^*(p) + G^*(p).$$

Linear Programing

Theorems 1.25-1.26 are applicable to the linear programing, where $X = \mathbf{R}^n$ and $(d,p) \in X \times X^*$ with $d,r \geq 0$. The last relation means that any components of d and r are non-negative.
In more precise, for a non-void closed convex set $K \subset \mathbf{R}^n$, its indicator function is denoted by

$$1_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & \text{otherwise.} \end{cases}$$

Then we define the skew-Lagrangian by

$$L(x,q) = -r\cdot x - q\cdot(d-Ax) - 1_{x\geq 0}(x) + 1_{q\geq 0}(q).$$

Since

$$\sup_{q\geq 0} z\cdot q = \begin{cases} 0, & z \leq 0 \\ +\infty, & \text{otherwise} \end{cases}$$

the cost function is defined by

$$\varphi(x,y) = \sup_q \{y \cdot q - L(x,q)\} = r \cdot x + 1_K(x,y)$$

for

$$K = \{(x,y) \mid x \geq 0,\ Ax \geq d + y\}$$

with the Legendre transformation

$$\begin{aligned}
\varphi^*(p,q) &= \sup_{x,y} \{x \cdot p + y \cdot q - \varphi(x,q)\} \\
&= \sup_{x\geq 0,\ y-Ax+d\leq 0} \{x \cdot (p-r) + (y - Ax + d) \cdot q + (Ax - d) \cdot q\} \\
&= 1_{q\geq 0}(q) - q \cdot d + \sup_{x\geq 0} (p - r + {}^tAq) \cdot x \\
&= \begin{cases} -qd, \text{ if } p - r + {}^tAq \leq 0,\ q \geq 0 \\ +\infty, \text{ otherwise.} \end{cases}
\end{aligned}$$

Thus, the principal and the dual problems are given by

$$\begin{aligned}
&(P) \ \text{ minimize } r \cdot x \text{ for } x \geq 0, Ax \geq d \\
&(P^*) \ \text{ maximize } q \cdot d \text{ for } q \geq 0, {}^tAq \leq r.
\end{aligned}$$

These problems have the same value from the duality theorem, and the solutions $\overline{x}$, $\overline{q}$ constitute of the saddle of L:

$$L(x,\overline{q}) \leq L(\overline{x},\overline{q}) \leq L(\overline{x},q)$$

for any (x,q).

Stokes System

The Stokes system

$$\begin{aligned}
&-\Delta u + \nabla p = f, \ \ \nabla \cdot u = 0 \qquad \text{in } \Omega \\
&u = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega
\end{aligned} \tag{1.206}$$

is identified with the saddle of the skew-Lagrangian

$$L(p,u) = \frac{1}{2}\|\nabla u\|_2^2 - (f,u) + (p, \nabla \cdot u) \tag{1.207}$$

defined for $(p,u) \in Y \times X$, where $X = H_0^1(\Omega)^n$, $Y = L^2(\Omega)$, and $(\cdot,\cdot)$ denotes the L^2-inner product. In this case, (1.206) is degenerate in p-component, regarded as a saddle $(\overline{u},\overline{p})$ of L,

$$L(p,\overline{u}) \leq L(\overline{p},\overline{u}) \leq L(\overline{p},u), \tag{1.208}$$

where $(p,u) \in Y \times X$. The weak form of (1.206) is defined by

$$\begin{aligned}&(p,u) \in Y \times X\\&(\nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v), \qquad v \in X\\&(\nabla \cdot u, q) = 0, \qquad\qquad\qquad\qquad q \in Y,\end{aligned}$$

and we obtain the following theorem.

Theorem 1.27. *The Stokes system (1.206) is equivalent to (1.208).*

Proof: Let $(\overline{p},\overline{u}) \in Y \times X$ be a (weak) solution to (1.206) and take $(p,u) \in Y \times X$ arbitrary. Then it follows that

$$L(\overline{p},\overline{u}) - L(p,\overline{u}) = (\overline{p} - p, \nabla \cdot \overline{u}) = 0$$

and

$$\begin{aligned}&L(\overline{p},u) - L(\overline{p},\overline{u})\\&\quad = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla \overline{u}\|_2^2 - (f, u-\overline{u}) + (\overline{p}, \nabla \cdot (u-\overline{u}))\\&\quad = (\nabla \overline{u}, \nabla(u-\overline{u})) - (f, u-\overline{u}) + (\overline{p}, \nabla \cdot (u-\overline{u})) + \frac{1}{2}\|\nabla(u-\overline{u})\|_2^2\\&\quad = \frac{1}{2}\|\nabla(u-\overline{u})\|_2^2 \geq 0.\end{aligned}$$

Let $(\overline{p},\overline{u}) \in Y \times X$ be a saddle of L, conversely. First, we have

$$L(\overline{p},\overline{u}) \geq L(p,\overline{u})$$

and hence $(\overline{p} - p, \nabla \overline{u}) \geq 0$ for any $p \in Y$ which implies

$$(\nabla \cdot \overline{u}, p) = 0$$

for any $p \in Y$. Next, we have

$$L(\overline{p},u) \geq L(\overline{p},\overline{u}), \qquad u \in X$$

which means that

$$(\nabla \overline{u}, \nabla w) - (f, w) + (\overline{p}, \nabla \cdot w) + \frac{1}{2}\|\nabla w\|_2^2 \geq 0, \qquad w \in X \tag{1.209}$$

and, therefore, we obtain

$$(\nabla \overline{u}, \nabla w) - (f, w) + (\overline{p}, \nabla \cdot w) \geq 0, \qquad w \in X,$$

checking the degree in w of (1.209). Then it follows that

$$(\nabla \overline{u}, \nabla w) - (f, w) + (\overline{p}, \nabla \cdot w) = 0, \qquad w \in X,$$

and $(\overline{p},\overline{u})$ is a weak solution to (1.206). □

The cost function $\varphi(p,u)$ associated with the skew Lagrangian $L(p,u)$ of (1.207) is defined for $(p,u) \in Y^* \times X$:

$$\begin{aligned}\varphi(p,u) &= \sup_{z\in Y}\{\langle z,p\rangle + L(z,u)\}\\ &= \sup_{z\in Y}\{\langle z,p\rangle + \frac{1}{2}\|\nabla u\|_2^2 - (f,u) + (\nabla\cdot u, z)\}\\ &= \frac{1}{2}\|\nabla v\|_2^2 - (f,v) + 1_{\{p=-\nabla\cdot u\}}(p),\end{aligned}$$

and, therefore,

$$\begin{aligned}\varphi^*(w,q) &= \sup_{v\in X}\{\langle v,w\rangle - L(q,v)\}\\ &= \sup_{p\in Y,\ v\in X}\{\langle v,w\rangle + (p,q) - \varphi(p,v)\}\\ &= \sup_{v\in X}\{\langle v,w\rangle - \frac{1}{2}\|\nabla v\|_2^2 + (f,v) - (q,\nabla\cdot v)\}\end{aligned}$$

for $(q,w) \in X^* \times Y$. The principal problem is thus given by

$$(P) \qquad \inf\{\frac{1}{2}\,\|\nabla v\|_2^2 - (f,v) \mid v \in X,\ \nabla\cdot v = 0\}\,.$$

Theorem 1.28. *The above (P) is equivalent to (1.206).*

Proof: Defining $B : H_0^1(\Omega)^n \to Y = L^2(\Omega)$ by

$$Bv = \nabla\cdot v,$$

we see that u solves (P) if and only if

$$\begin{aligned}&u \in \operatorname{Ker} B \subset X\\ &(\nabla u, \nabla v) = (f,v), \qquad v \in \operatorname{Ker} B.\end{aligned} \tag{1.210}$$

If $\sigma : X \to X^*$ denotes the *duality map* defined by

$$\langle \sigma f, v\rangle_{X^*,\,X} = (f,v), \qquad v \in X,$$

then, (1.210) means

$$\begin{aligned}&u \in \operatorname{Ker} B\\ &u - \sigma f \in (\operatorname{Ker} B)^\perp = \overline{\operatorname{Ran} B^*}\end{aligned}$$

The Poincaré inequality, on the other hand, implies

$$\|\nabla v\|_2^2 \geq \delta\|v\|_2^2, \qquad v \in H_0^1(\Omega)^n,\ \nabla\cdot v = 0$$

and, therefore, $\operatorname{Ran} B^*$ is closed in X^*. Thus (p,u) solves (P) if and only if

$$\begin{aligned}&u \in \operatorname{Ker} B \subset X\\ &u - \sigma f = B^* p\end{aligned}$$

for some $p \in Y$, and hence (1.206). □

In the abstract formulation, we take

$$L(y,x) = F(x) + (y,Bx)_Y$$

for $(y,x) \in Y \times X$, where X and Y are Banach and Hilbert spaces over $\mathbf{R}$, $F : X \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous, and $B : X \to Y$ is bounded linear with $\mathrm{Ran}(B^*)$ closed in X^*, see[106] for more details.

1.3.6. Summary

We have examined several mathematical models concerning self-assembly, which are provided with the variational structure derived from the free energy or (skew) Lagrangian.

(1) Closed Systems

  (a) Model (A) equation describes thermally closed system.
  (b) Model (B) equation describes thermally and materially closed system.

(2) (Skew) Gradient Systems

  (a) Several model (C) equations are formulated as (skew) gradient flows derived from the (skew) Lagrangian. A typical example is the gradient system provided with the Toland duality.
  (b) If the skew-Lagrangian is provided with a duality, then the dynamics and the structure of the stationary solutions are simple.
  (c) Skew-gradient system is involved by the dissipative structure, but the dynamics around the stationary solution is controlled by the calculus of variation.

(3) Non-convex evolution

  (a) Several free energies are involved by the double-well potential. Then, the associated (skew-) Lagrangian is provided with the semi-duality.
  (b) If the (skew)-Lagrangian is provided with semi-duality, then the linearized stability, the robust stability, and the hyper-linear stability of the critial point are defined.
  (c) A hyper-linearly stable critical point is dynamically stable in the gradient system with semi-Lagrangian. This property is a case of the skew-gradient system with semi-skew-Lagrangian, while a linearly but not hyper-linearly stable semi-crtical point casts the driving force of the far-from-equilibrium.

1.4. Thermodynamics

Having examined model (A) and model (B) equations in the previous section, this section is devoted to the physical principles that derive model (C) equations consistent with the non-equilibrium thermodynamics. Several systems of phenomenological equations are thus derived from the free energy and duality. Similar to entropy, free energy is a fundamental concept of thermodynamics in accordance with the thermal equilibrium. This section is devoted to the theory of non-equilibrium thermodynamics from the viewpoint of dynamical systems.

1.4.1. Stefan Problem

We begin with the two-phased Stefan problem and its mathematical structures by examining first its free-boundary and interface problem, secondly, the reduction to a degenerate parabolic equation associated with the enthalpy, third, the application of the nonlinear semigroup theory to this equation, and finally, the phase field approach of the Ginzburg-Landau theory using order parameter.

Level Set Approach

Let θ be the relative temparature, and assume that the heat conductor is water and ice if $\theta > 0$ and $\theta < 0$, respectively. Then we obtain the heat equation (1.7),

$$c\rho\theta_t = \nabla\cdot(\kappa\nabla\theta) \qquad \text{in } \{\theta \neq 0\}. \tag{1.211}$$

The region $\{x \in \Omega \mid \theta(x,t) = 0\}$ is comprised of an interface, denoted by Γ_t, where the phase transition occurs and the latent heat is exchanged. Usually, the density $\rho = \rho(\theta)$ depends continuously on $\theta \in \mathbf{R}$, but the specific heat $c = c(\theta)$ and the conductivity $\kappa = \kappa(\theta)$ have the discontinuity of the first kind at $\theta = 0$.
To describe the motion of Γ_t, we take first the level-set approach, and introduce the C^1 function $\Phi = \Phi(x,t)$ satisfying

$$\Gamma_t : \Phi(\cdot,t) = 0. \tag{1.212}$$

Let ν be the outer unit normal vector of Γ_t from $\{\theta < 0\}$ at $x \in \Gamma_t$. If x moves $\nu\Delta h$ during the small time interval Δt, then it holds that

$$\Phi(x+\nu\Delta h, t+\Delta t) = 0.$$

Taking the infinitesimal approximation of this relation, we obtain

$$(\nu\cdot\nabla\Phi)\Delta h + \Phi_t\Delta t = 0 \qquad \text{on } \Gamma_t. \tag{1.213}$$

Meanwhile, $\ell\Delta h$ is radiated from the unit area on Γ_t as the latent heat, where $\ell = \lambda\rho$ with λ standing for the latent heat per unit weight, and, therefore, Newton-Fourier-Fick's heat energy balance law guarantees the relation

$$\ell\Delta h = -\left[\kappa\frac{\partial\theta}{\partial\nu}\right]_-^+ \Delta t, \tag{1.214}$$

where

$$[A]_-^+ = A_+ - A_-$$
$$A_\pm(x) = \lim_{y\in\{\pm\theta>0\},\ y\to x} A(y).$$

Combining (1.213) with (1.214), thus, we obtain the Stefan condition

$$\ell\Phi_t = \frac{\partial\Phi}{\partial\nu}\cdot\left[\kappa\frac{\partial\theta}{\partial\nu}\right]_-^+ \qquad \text{on } \Gamma_t, \tag{1.215}$$

which comprises of the Stefan problem with (1.211) and (1.212).

Enthalpy Formulation

Here, we use the Kirchhoff transformation

$$u = \int_0^\theta \kappa(\theta')d\theta'$$

and the enthalpy $H = H(u)$ defined by

$$H(u) = \begin{cases} \int_0^\theta \rho(\theta')c(\theta')d\theta' - \ell, & u < 0 \\ \int_0^\theta \rho(\theta')c(\theta')d\theta', & u > 0, \end{cases}$$

which satisfies

$$\begin{aligned} H'(u) &= \frac{\rho(\theta)c(\theta)}{\kappa(\theta)}, \qquad u \neq 0 \\ H(+0) - H(-0) &= \ell. \end{aligned}$$

Concerning $u = u(x,t)$, we obtain

$$\begin{aligned} \nabla u &= \kappa \nabla \theta \\ H(u)_t &= \frac{\rho c}{\kappa} \cdot \theta_t \cdot \kappa \end{aligned}$$

and, therefore,

$$\begin{aligned} H(u)_t &= \Delta u && \text{in } \bigcup_{0<t<T} (\Omega \setminus \Gamma_t) \times \{t\} \\ \ell \Phi_t &= [\nabla u \cdot \nabla \Phi]_-^+ && \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\} \end{aligned} \tag{1.216}$$

by (1.211) and (1.214).
Now, we show

$$\frac{d}{dt}\int_\omega H(u)dx = \int_{\partial\omega} \frac{\partial u}{\partial \nu} dS \tag{1.217}$$

for each $\omega \subset \Omega$ with C^1 boundary $\partial\omega$. In fact, we can assume $\omega \cap \Gamma_t \neq \emptyset$ by (1.216) and, furthermore, that this portion is a cylinder with the small base area σ on Γ_t and the height $2\Delta h$ cut by $S_{t\pm\Delta t}$ for $0 < \Delta t \ll 1$, similarly to the proof of the divergence formula of Gauss (§1.1.1). If ν denotes the outer unit vector on Γ_t from $\{\theta < 0\}$, then (1.215) means

$$\ell \frac{\Delta h}{\Delta t} = -\left[\frac{\partial u}{\partial \nu}\right]_-^+ \tag{1.218}$$

similarly to (1.214). Relation (1.217), on the other hand, implies

$$\int_\omega H(u)dx \bigg|_{t-\Delta t}^{t+\Delta t} = \int_{t-\Delta t}^{t+\Delta t} dt \cdot \int_\omega \frac{\partial u}{\partial \nu} dS. \tag{1.219}$$

Here, the right-hand and left-hand sides are identified with

$$\begin{aligned} \int_\omega [H(u(x,t+\Delta t)) - H(u(x,t-\Delta t))]\,dx &= 2\Delta h \cdot \sigma \cdot [H(u)_- - H(u)_+] \\ &= -2\Delta h \cdot \sigma \cdot \ell, \end{aligned}$$

and

$$2\Delta t \cdot \sigma \cdot \left[\frac{\partial u}{\partial \nu}\right]_-^+,$$

respectively, and, therefore, it holds that (1.217).
Using the divergence formula (1.1) again, we obtain

$$H(u)_t = \Delta u \qquad \text{in } \Omega \times (0,T) \tag{1.220}$$

by (1.217), which re-formulates the Stefan problem with the initial-boudary condition

$$\begin{aligned} u|_{t=0} &= u_0(x) \\ u|_{\partial\Omega} &= g(\xi,t). \end{aligned} \tag{1.221}$$

Here, $H = H(u)$ takes the discontinuity of the first kind at $u = 0$. If ρ is constant and c and κ are piecewise constant, then it holds that

$$\begin{aligned} H'(u) &= \begin{cases} c_+\kappa_+, & u > 0 \\ c_-\kappa_-, & u < 0 \end{cases} \\ H(+0) - H(-0) &= \ell. \end{aligned}$$

In any case, we define the maximum monotone graph, still denoted by $H = H(u)$ in $\mathbf{R} \times \mathbf{R}$, putting $H(0) = [H(-0), H(+0)]$. Then we obtain the unique existence of the weak solution

$$u = u(x,t) \in L^\infty(\Omega \times (0,T))$$

to (1.220)-(1.221) globally in time.[160,237] More precisely, there is $h = h(x,t)$ such that $h \in H(u)$ a.e. and

$$\iint_Q (u\Delta\varphi + h\varphi_t)dxdt - \iint_\Gamma g\frac{\partial\varphi}{\partial\nu}dSdt + \int_\Omega u_0\varphi(\cdot,0)dx = 0$$

for any $\varphi = \varphi(x,t) \in C^{2,1}(\overline{\Omega} \times [0,T])$ with $\varphi|_{t=T} = 0$, where

$$\begin{aligned} Q &= \Omega \times (0,T) \\ \Gamma &= \partial\Omega \times (0,T). \end{aligned}$$

Nonlinear Semi-Group Theory

Using the single-valued maximal monotone graph $f(v) = H^{-1}(v)$, we can derive

$$v_t = \Delta f(v) \qquad \text{in } \Omega \times (0,t)$$

from (1.220), and then the nonlinear semi-group theory[16,202] is applicable. In more detail, if $f : \mathbf{R} \to \mathbf{R}$ is a non-decreasing continuous function satisfying $f(0) = 0$, then the operator $Av = -\Delta f(v)$ with the domain

$$D(A) = \left\{ v \in L^1(\Omega) \mid f(v) \in W_0^{1,1}(\Omega),\ \ \Delta f(v) \in L^1(\Omega) \right\}$$

is maximum monotone,[33] and hence generates a contraction semi-group[69] in $X = L^1(\Omega)$, detoted by $\{T_t\}_{t\geq 0}$. Then $v(\cdot,t) = T_t v_0$ is regarded as a solution to

$$\begin{aligned} v_t &= \Delta f(v) \qquad \text{in } \Omega \times (0,T) \\ v|_{t=0} &= v_0 \qquad \text{in } \Omega, \end{aligned} \tag{1.222}$$

and in this sense the problem (1.222) is well-posed in $X = L^1(\Omega)$ globally in time.

Phase Field Model

Classical Stefan problem is the origin of the phase field theory. First, assuming that the physical coefficients are piecewise constant for simplicity, we replace (1.220) by

$$H(u)_t = \kappa \Delta u \qquad \text{in } \Omega \times (0,T), \tag{1.223}$$

where

$$H(u) = u + \frac{\ell}{2}\varphi$$
$$\varphi = \begin{cases} +1, & u > 0 \\ -1, & u < 0. \end{cases}$$

This φ is regarded as the phase, which is equal to $+1$ and -1 if the material is water and ice, respectively.

In the Ginzburg-Landau theory, however, this phase is replaced by the order parameter which varies continuously in space and time. This order parameter is controlled by the free energy $\mathscr{F}(\varphi)$, indicating a macroscopic quantity describing the microscopic freedom of the system. In other words, the system retains high freedom of microscopic states when the value of this free energy is high, and it becomes a minimum if the system falls in the equilibrium. Thus, the system passages to the state with the local minimum free energy, and model (A) or model (B) equation defined in §1.3.1 realizes such a principle.

If $\varphi = \varphi(x)$ is subject to Ginzburg-Landau's free energy (1.180),

$$\mathscr{F}(\varphi) = \int_\Omega \frac{\xi^2}{2} |\nabla \varphi|^2 + W(\varphi)\, dx,$$

then the phase of this system is almost water and ice if $\varphi \geq 1$ and $\varphi \leq -1$, respectively, while *mushy region* is defined by

$$\{(x,t) \in \Omega \times (0,T) \mid -1 < \varphi(x,t) < +1\}.$$

In the Fix-Caginalp equation,[38,65,99,178] the above free energy is modified in accordance with u, regarded as a relative temperature, that is

$$\mathscr{F}_u(\varphi) = \int_\Omega \frac{\xi^2}{2} |\nabla \varphi|^2 + W(\varphi) - 2u\varphi\, dx,$$

and, therefore, this system is provided with the high free energy if $u\varphi < 0$, and passages to $u = 0$ with $\varphi = \pm 1$ eventually. Then we obtain

$$\begin{aligned} &\alpha\varphi_t = \xi^2\Delta\varphi + (\varphi - \varphi^3) + 2u \\ &u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta\varphi \qquad \text{in } \Omega \times (0,T), \end{aligned} \tag{1.224}$$

using model (A) equation for φ and the entalpy equation (1.223) with

$$H = u + \frac{\ell}{2}\varphi.$$

In the classical theory of thermodynamics, however, decrease of the free energy occurs under the constant temperature, and in this sense this model is not valid if u is far from zero.

1.4.2. Thermal Equilibrium

This paragraph is devoted to a review of equilibrium thermodynamics, and clarify what is described at the end of the previous paragraph.

Quantity of State

First, we confirm that the objects of thermodynamics are the quantity of state, independent of the history of the system. The (absolute) temperature T, the pressure p, and the volume V are the quantities of state of the ideal gas, subject to the state equation

$$f(p,T,V) = 0.$$

A thermodynamical quantity of state A, therefore, is a function of two variables of p, T, and V, denoted by x and y, satisfying

$$\int_\gamma dA = 0$$

for any closed path γ in the xy plane, where

$$dA = Xdx + Ydy$$

with

$$X = \frac{\partial A}{\partial x}, \qquad Y = \frac{\partial A}{\partial y}.$$

Thus, this one-form $Xdx + Ydy$ is completely integrable, and it holds that

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

Energy Balance

The first law of thermodynamics is the energy balance described by

$$dU = Q - pdV,$$

where the left-hand side indicates the energy variation of the system in accordance with the volume variation in the right-hand side caused by the outer thermal energy Q. Here, it is emphasized that heat or work is not a quantity state.

Entropy Increasing

The second law of thermodynamics is the entropy increasing. If the process is reversible and $d'Q$ denotes the energy variation, then we can examine

$$\int_\gamma \frac{d'Q}{T} = 0$$

for any closed path γ, and thus we can define the entropy variation dS by

$$dS = \frac{d'Q}{T}$$

in this case. This relation allows us to re-fomulate entropy as a state quanity valid even in the irreversible system. Then, the Clausius-Duhem's inequality

$$dS > \frac{d'Q}{T} \tag{1.225}$$

is obtained if the process is irreversible.
Entropy decomposition is obtained by introducing the inside thermal energy transport $d'Q^*$ caused by the thermal energy contact, denoted by $d'Q_{ir}$, to the outer system. More precisely, the entropy variation and the inner entropy production arising at this contact are defined by

$$d_eS = \frac{d'Q_{ir}}{T}$$

and

$$d_iS = \frac{d'Q^*}{T},$$

respectively, and then inequality (1.225) is replaced by the equality

$$dS = d_eS + d_iS.$$

Thus, the second law of thermodynamics is re-formulated by $d_iS \geq 0$ with the equality if and only if the process is reversible.

Free Energy

The first and second laws of thermodynamics are thus summarized by

$$\begin{aligned} dU &= d'Q - pdV \\ dS &= \frac{d'Q}{T} + d_iS. \end{aligned} \tag{1.226}$$

Then, Helmholtz' free energy A, Gibbs' free energy G, and the enthalpy H are defined by

$$A = U - TS$$
$$G = H - TS$$
$$H = U + pV.$$

If the process is iso-thermal and iso-volumetric, then it holds that

$$dA = dU - TdS = -Td_iS \leq 0$$

with the equality if and only if it is reversible. Similarly, if the process is iso-thermal and iso-baric, then it holds that

$$dG = dH - TdS = dU + pdV - TdS = -Td_iS \leq 0$$

with the equality if and only if it is reversible.
In the reversible process of $d_iS = 0$, it follows that

$$dU = TdS - pdV,$$

which guarantees

$$\begin{aligned} dH &= TdS + Vdp \\ dA &= -SdT - pdV \\ dG &= -SdT + Vdp \end{aligned} \tag{1.227}$$

and, therefore,

$$\begin{aligned} T &= \left(\frac{\partial U}{\partial S}\right)_V, \quad p = -\left(\frac{\partial U}{\partial V}\right)_S \\ T &= \left(\frac{\partial H}{\partial S}\right)_p, \quad V = \left(\frac{\partial H}{\partial p}\right)_S \\ S &= -\left(\frac{\partial A}{\partial T}\right)_V, \quad p = -\left(\frac{\partial A}{\partial V}\right)_T \\ S &= -\left(\frac{\partial G}{\partial T}\right)_p, \quad V = -\left(\frac{\partial G}{\partial p}\right)_T. \end{aligned}$$

These equalities result in

$$\begin{aligned} \left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial p}{\partial S}\right)_V, \quad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p \\ \left(\frac{\partial S}{\partial V}\right)_T &= \left(\frac{\partial p}{\partial T}\right)_V, \quad \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p, \end{aligned}$$

called Maxwell's relation. Here and henceforth, $*$ in $(\ )_*$ indicates the fixed variable in taking derivatives.

Chemical Potential

In the open system provided with the material transport between the outer system, Gibbs' free energy is a function of the numbers of moleculars comprising the following system

$$G = G(T, p, n_1, n_2, \ldots).$$

In this case, dG of (1.227) shifts to

$$dG = -SdT + Vdp + \sum_i \mu_i dn_i,$$

where

$$\mu_i = \left(\frac{\partial G}{\partial n_i}\right)_{T,p,n_{j \neq i}}$$

is called the chemical potential. It is assumed to be a partial molal quantity, which means

$$\mu_i = \left(\frac{\partial U}{\partial n_i}\right)_{S,V,n_{j \neq i}} = \left(\frac{\partial H}{\partial n_i}\right)_{S,p,n_{j \neq i}} = \left(\frac{\partial A}{\partial n_i}\right)_{T,V,n_{j \neq i}},$$

or, equivalently,

$$\begin{aligned} dU &= TdS - pdV + \sum_i \mu_i dn_i \\ dH &= TdS + Vdp + \sum_i \mu_i dn_i \\ dA &= -SdT - pdV + \sum_i \mu_i dn_i. \end{aligned} \tag{1.228}$$

If the system is iso-thermal and iso-baric, then we obtain

$$dG = \sum_i \mu_i dn_i. \tag{1.229}$$

Therefore, if the material composition is constant, furthermore, then it follows that

$$G = \sum_i \mu_i n_i$$

and hence

$$\sum_i n_i d\mu_i = 0.$$

It is called Gibbs-Duhem's equality.
If ξ denotes the extent of reaction, then

$$dn_i = \nu_i d\xi$$

is the variance of the i-component of the material, when the chemical reaction proceeds from ξ to $\xi + d\xi$, where ν_i denotes the stoichiometric coefficient. Its time variance is defined by

$$\frac{dn_i}{dt} = \nu_i \frac{d\xi}{dt},$$

or

$$\frac{dn_i}{dt} = \sum_k \nu_{ik} \frac{d\xi_k}{dt} \tag{1.230}$$

if this component is associated with the other reactions.

Equilibrium

If two interacting systems A and B are in equilibrium and are isolated totally, then the total entropy, the total inner energy, the total volume, and the total quatntity of i component are constant.

$$S_A + S_B = \text{constant}, \qquad U_A + U_B = \text{constant}$$
$$V_A + V_B = \text{constant}, \qquad n_{iA} + n_{iB} = \text{constant}.$$

From $dS_A + dS_B = 0$, it follows that

$$\left(\frac{dU_A}{T_A} + \frac{p_A}{T_A} dV_A - \sum_i \frac{\mu_{iA}}{T_A} dn_{iA}\right) + \left(\frac{dU_B}{T_B} + \frac{p_B}{T_B} dV_B - \sum_i \frac{\mu_{iB}}{T_B} dn_{iB}\right) = 0.$$

Then, using

$$dU_A + dU_B = 0$$
$$dV_A + dV_B = 0$$
$$dn_{iA} + dn_{iB} = 0,$$

we obtain

$$\left(\frac{1}{T_A} - \frac{1}{T_B}\right) dU_A + \left(\frac{p_A}{T_A} - \frac{p_B}{T_B}\right) dV_A - \sum_i \left(\frac{\mu_{iA}}{T_A} - \frac{\mu_{iB}}{T_B}\right) dn_{iA} = 0,$$

which results in

$$T_A = T_B$$
$$p_A = p_B$$
$$\mu_{iA} = \mu_{iB}.$$

Thus, the equilibrium of this contact is described by the balance of the temperature, the pressure, and the chemical potential of two systems.
In the iso-thermal and iso-baric system, we have (1.229). In the chemical reaction system

$$aA + bB \rightleftharpoons cC + dD,$$

for example, the free energy variation is given by

$$\Delta G = c\mu_C + d\mu_D - a\mu_A - b\mu_B.$$

The chemical potential of ideal solution, on the other hand, is equal to

$$\mu_i = \mu_i^0 + RT \log c_i, \tag{1.231}$$

where R, c_i, and μ_i^0 are the gas constant, the concentration, and the normal chemical potential, respectively. Thus in the equilibrium of $\Delta G = 0$, it holds that

$$\Delta G^0 = -RT\log K$$
$$K = \frac{[C]_{eq}^c [D]_{eq}^d}{[A]_{eq}^a [B]_{eq}^b}$$
$$G^0 = c\mu_c^0 + d\mu_D^0 - a\mu_A^0 - b\mu_B^0,$$

where $[\,]_{eq}$ denotes the equilibrium concentration and K is called the equilibrium coefficient of reaction.
In the chemical reaction of

$$\nu_1 R_1 + \nu_2 R_2 + \ldots + \nu_j R_j \to \nu_{j+1} p_1 + \nu_{j+2} p_2 + \ldots \nu_c p_c,$$

we replace $-\nu_i$ by ν_i for $i \geq j+1$, and define the chemical affinity by

$$A = -\sum_{i=1}^{j} \nu_i \mu_i.$$

The inner entropy production of this process is given by

$$d'Q^* = Td_iS = -dG = -\sum_i \mu_i dn_i = Ad\xi \tag{1.232}$$

from $\nu_i d\xi = dn_i$.

1.4.3. Penrose-Fife Theory

If the temperature varies, it is preferable to use the equation provided with the increase of entropy other than the decrease of free energy. This property is realized by the Penrose-Fife theory,[252] and then the Penrose-Fife and the coupled Cahn-Hilliard equations are obtained for the phase transition and the phase separation, respectively.
First, we infer

$$f(T) = \inf_e (e - Ts(e)) \tag{1.233}$$

from $A = U - TS$ and the minimum energy principle, where e, $s(e)$, and $f(T)$ are the energy density, the entropy density, and the free energy density, respectively. Since $e \mapsto s(e)$ is concave, this (1.233) is regarded as the Legendre transformation. The minimum is attained by $e = e(T)$ satisfying

$$\frac{\partial s}{\partial e} = \frac{1}{T}, \tag{1.234}$$

and hence it holds that

$$\frac{f(T)}{T} = \frac{e(T)}{T} - s(e(T)).$$

Using (1.234), we obtain

$$\frac{d(f(T)/T)}{d(1/T)} = e(T) + \frac{1}{T}\frac{de(T)}{d(1/T)} - \frac{\partial s}{\partial e} \cdot \frac{de(T)}{d(1/T)} = e(T).$$

Thus, the minimum of (1.233) is attained by

$$e = \frac{d(f(T)/T)}{d(1/T)}, \tag{1.235}$$

and, therefore, it holds that

$$e = f - Tf_T. \tag{1.236}$$

Writing (1.233) as

$$\frac{f(T)}{T} = \inf_e \left(e \cdot \frac{1}{T} - s(e) \right),$$

we obtain

$$s(e) = \inf_T \left\{ \frac{e}{T} - \frac{f(T)}{T} \right\} \tag{1.237}$$

by Fenchel-Moreau's duality. Furthermore, the minimum of (1.237) is attained by $T = T(e)$, that is, the inverse function of $e = e(T)$ defined by (1.234). These relations are valid even when f depends on the order parameter φ, and we can define the entropy s by (1.237), namely,

$$\begin{aligned} f(T,\varphi) &= \inf_e \{e - Ts(e,\varphi)\} \\ s(e,\varphi) &= \inf_T \left\{ \frac{e}{T} - \frac{f(T,\varphi)}{T} \right\}. \end{aligned} \tag{1.238}$$

If $T = T(e,\varphi)$ attains the second minimum of (1.238), then it holds that

$$\frac{\partial s(e,\varphi)}{\partial e} = \frac{1}{T(e,\varphi)} \tag{1.239}$$

similarly to (1.234). Here, we apply the transformation of variables

$$(e,\varphi) \mapsto (1/T,\varphi)$$

for

$$\begin{aligned} s(e,\varphi) &= \frac{e}{T} - \frac{f}{T} \\ f &= f(T,\varphi) \\ T &= T(e,\varphi), \end{aligned}$$

and obtain

$$\begin{aligned} \frac{\partial s(e,\varphi)}{\partial \varphi} &= e \cdot \frac{\partial}{\partial \varphi}\left(\frac{1}{T}\right) - \left\{ \frac{\partial (f/T)}{\partial (1/T)} \cdot \frac{\partial (1/T)}{\partial \varphi} + \frac{1}{T}\frac{\partial f}{\partial \varphi} \right\} \\ &= -\frac{1}{T(e,\varphi)} \frac{\partial f(T(e,\varphi),\varphi)}{\partial \varphi}. \end{aligned} \tag{1.240}$$

Using $\varphi = \varphi(x)$, the free energy functional is now defined by

$$\begin{aligned}\mathscr{F}(T,\varphi) &= \int_\Omega f(T,\varphi(x)) + \frac{\xi^2}{2}|\nabla\varphi(x)|^2\, dx \\ &= \int_\Omega \inf_e [e - Ts(e,\varphi(x))] + \frac{\xi^2}{2}|\nabla\varphi(x)|^2\, dx \\ &= \inf_{e(\cdot)} \int_\Omega e(x) - Ts(e(x),\varphi(x)) + \frac{\xi^2}{2}|\nabla\varphi(x)|^2\, dx \\ &= \inf_{e(\cdot)} \{U(e) - TS(e,\varphi)\},\end{aligned}$$

where

$$U(e) = \int_\Omega e(x)dx$$

and

$$S(e,\varphi) = \int_\Omega s(e(x),\varphi(x)) - \frac{\kappa_1}{2}|\nabla\varphi(x)|^2\, dx$$

are the energy and the entropy functionals, respectively, and $\kappa_1 = \xi^2/T$. Here, taking variations of this entropy functional, we obtain

$$\frac{\delta S(e,\varphi)}{\delta e} = \frac{\partial s}{\partial e}(e(x),\varphi(x)) = \frac{1}{T(e,\varphi)} \tag{1.241}$$

and

$$\begin{aligned}\frac{\delta S(e,\varphi)}{\delta\varphi} &= \frac{\partial s}{\partial\varphi}(e(x),\varphi(x)) + \frac{\xi^2}{T(e(x),\varphi(x))}\Delta\varphi(x) \\ &= \frac{1}{T(e,\varphi)}\left(-\frac{\partial f}{\partial\varphi}(T(e,\varphi),\varphi) + \xi^2\Delta\varphi\right) \\ &= -\frac{1}{T}\frac{\partial f}{\partial e} + \kappa_1\Delta\varphi,\end{aligned} \tag{1.242}$$

by (1.239) and (1.240), respectively.

Concerning the time evolution of the order parameter, model (A) equation is adopted by the entropy increasing,

$$\varphi_t = K_1\frac{\delta S}{\delta\varphi}(e,\varphi). \tag{1.243}$$

On the other hand, model (B) equation is adopted for the energy density evolution by the entropy increasing and the energy conservation,

$$e_t = -\nabla\cdot\left(M_2\nabla\frac{\delta S}{\delta e}(e,\varphi)\right). \tag{1.244}$$

Finally, we assume the energy density by

$$e = u(T)\nu(\varphi) + \lambda(\varphi),$$

where $\lambda(\varphi)$ is a potential, and

$$u(T) = \frac{1}{2}kT$$

with the Boltzmann constant k and the degree of microscopic state freedom $\nu(\varphi)$ per unit volume.[252] In this case, the first term of e indicates the kinetic energy, and it holds that

$$\begin{aligned} f(T,\varphi) &= \nu(\varphi)u_1(T) + \lambda(\varphi) - Ts_0(\varphi) \\ u_1(T) &= T\int u(T)d(1/T) \end{aligned} \tag{1.245}$$

with the integration constant $s_0(\varphi)$. Thus, we obtain

$$\begin{aligned} s(e,\varphi) &= \nu(\varphi)y(T(e,\varphi)) + s_0(\varphi) \\ y(T) &= \frac{u(T)-u_1(T)}{T} = \int \frac{du}{T} \end{aligned}$$

by (1.238), and then (1.243)-(1.244) reads as follows.

$$\begin{aligned} \varphi_t &= K_1\left(-\frac{u_1(T)\nu'(\varphi)}{T} - \frac{\lambda'(\varphi)}{T} + s_0'(\varphi) + \kappa_1\Delta\varphi\right) \\ u_t\nu(\varphi) + u\nu'(\varphi)\varphi_t + \lambda'(\varphi)\varphi_t &= -\nabla\cdot\left(M_2\nabla\frac{1}{T}\right). \end{aligned} \tag{1.246}$$

Sometimes $\nu(\varphi)$ and $s_0(\varphi)$ are assumed to be a constant and $\frac{1}{4}\left(\varphi^2-1\right)^2$, respectively. In this case, it follows that

$$\begin{aligned} \varphi_t &= K_1\left(\kappa_1\Delta\varphi + \varphi - \varphi^3 - T^{-1}\lambda'(\varphi)\right) \\ \frac{k}{2}T_t + \lambda(\varphi)\varphi_t &= -\nabla\cdot\left(M_2\nabla\left(\frac{1}{T}\right)\right). \end{aligned} \tag{1.247}$$

Alt-Pawlow[6,7] obtained this equation using the renormalized free energy density $\tilde{f}(e,T) = f(e,T)/T$, where the heat flux $q = \kappa(e,T)\nabla(1/T)$ is defined by Newton-Fourier-Fick's law concerning the energy balance, and the increase of entropy is derived from Clausius-Duhem's inequality.[34]

1.4.4. Non-Equilibrium

Although the system of balance laws can describe open systems, local equilibrium principle guarantees the Prigogine paradigm near the stable equilibrium, whereby the non-equilibrium stationary state is defined. In the range where the *phenomenological relation* is valid, this stationary state thus realizes the minimum entropy production rate. Then the *detailed balance law* implies *Onsager's reciprocity*, and the irreducible process is identified with null dissipation.[75,224] A useful application of this theory is the understanding of motion of the free boundary associated with a phase transition.[329]

Balance Laws

First, local equilibrium principle assures that the entropy, the inner energy, and the material desnity per unit volume v are state quantities, denoted by $s_v = s/v$, $u_v = u/v$, and $c_i = n_i/v$, $1 \le i \le n$, respectively. Their balance laws are described by

$$\frac{\partial s_v}{\partial t} = -\nabla \cdot J_s + \sigma$$
$$\frac{\partial c_i}{\partial t} = -\nabla \cdot J_i + \nu_i J_{ch}$$
$$\frac{\partial q_v}{\partial t} = -\nabla \cdot J_q. \tag{1.248}$$

Here, the first equation of (1.248) indicates the entropy balance, where J_s and σ are the entropy flux and the local entropy production rate, respectively, and hence it holds that

$$\frac{d_i S}{dt} = \int \sigma dV.$$

In the second equation, J_i and $J_{ch} = \frac{d\xi}{dt}$ are the diffusion flux and the chemical reaction rate, respectively, and, therefore,

$$\nu_i J_{ch} = \frac{dn_i}{dt}$$

stands for the production rate of the chemical material. In the third equation, J_q and q_v denote the total heat flux and the local heat quantity, respectively.

Dissipation Function

Above mentioned local equilibrium principle induces

$$T ds_v = du_v - \sum_i \mu_i dc_i$$

by (1.228), if the process is iso-volumetric. It also holds that $du_v = dq_v$ and $dU = d'Q$ by (1.226), and hence

$$\frac{\partial s_v}{\partial t} = \frac{1}{T}\frac{\partial q_v}{\partial t} - \sum_i \frac{\mu_i}{T}\frac{\partial C_i}{\partial t}.$$

Using (1.248), we obtain

$$-\nabla \cdot J_s + \sigma = -\frac{1}{T}\nabla \cdot J_q - \sum_i \frac{\mu_i}{T}(-\nabla \cdot J_i + \nu_i J_{ch}).$$

Here, in the right-hand side, we have

$$\frac{1}{T}\nabla \cdot J_q = \nabla \cdot \frac{J_q}{T} - J_q \cdot \nabla \frac{1}{T}$$
$$\frac{\mu_i}{T}\nabla \cdot J_i = \nabla \cdot \frac{\mu_i J_i}{T} - J_i \cdot \nabla \frac{\mu_i}{T},$$

and, therefore,

$$
\begin{aligned}
-\nabla \cdot J_s + \sigma &= -\nabla \cdot \frac{J_q - \sum_i \mu_i J_i}{T} \\
&\quad + J_q \cdot \nabla \frac{1}{T} + \sum_i J_i \cdot \nabla \left(-\frac{\mu_i}{T} \right) + J_{ch} \frac{A}{T},
\end{aligned} \tag{1.249}
$$

where $A = -\sum_i \nu_i \mu_i$ is the chemical affinity.
Taking regards to (1.249), we put

$$
\begin{aligned}
J_s &= \frac{J_q - \sum_i \mu_i J_i}{T} \\
\sigma &= J_q \cdot \nabla \frac{1}{T} + \sum_i J_i \nabla \left(-\frac{\mu_i}{T} \right) + J_{ch} \frac{A}{T}.
\end{aligned}
$$

Then, using

$$
\begin{aligned}
\nabla \frac{1}{T} &= -\frac{1}{T^2} \nabla T \\
\nabla \left(-\frac{\mu_i}{T} \right) &= -\frac{1}{T} \nabla \mu_i + \frac{\mu_i}{T^2} \nabla T,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\sigma &= -\frac{J_q - \sum_i \mu_i J_i}{T^2} \cdot \nabla T - \sum_i \frac{J_i}{T} \cdot \nabla \mu_i + J_{ch} \frac{A}{T} \\
&= \frac{J_s}{T} \cdot \nabla(-T) + \sum_i \frac{J_i}{T} \cdot \nabla(-\mu_i) + J_{ch} \frac{A}{T}.
\end{aligned}
$$

This equality allows us to define the dissipation function

$$
\Phi = T\sigma = J_s \cdot \nabla(-T) + \sum_i J_i \cdot \nabla(-\mu_i) + J_{ch} A = \sum_{i=0}^{n+1} J_i \cdot X_i,
$$

where J_i and X_i are regarded as the flux and the general force, respectively. In more detail, $J_0 = J_s$ is the entropy flux subject to $X_0 = -\nabla T$ with the temperature T, J_i, $i \le n$, is the i-th diffusion flux subject to $X_i = -\nabla \mu_i$ with the chemical potential μ_i, and $X_{n+1} = J_{ch}$ is the chemical velocity subject to the chemical affinity $X_{n+1} = A$.

Entropy Production Rate

In the iso-thermal system with diffusion and chemical reaction, we can define the entropy production rate by

$$
P = T \frac{d_i S}{dt} = \int \Phi dV \ge 0 \tag{1.250}
$$

using the dissipation function. Here, the equality means the equilibrium of the system. We have, on the other hand,

$$
\Phi = -\sum_{i=1}^{n} J_i \cdot \nabla \mu_i + \sum_k A_k v_k
$$

in this case, and hence it holds that

$$\frac{dP}{dt} = \int \sum_{i=1}^{n+1} J_i \frac{\partial X_i}{\partial t} + \sum_{i=1}^{n+1} X_i \frac{\partial J_i}{\partial t} \, dV$$
$$\equiv \frac{d_X P}{dt} + \frac{d_J P}{dt}.$$

Using (1.230), then we obtain

$$\frac{d_X P}{dt} = \int -\sum_i J_i \cdot \nabla \frac{\partial \mu_i}{\partial t} + \sum_k v_k \frac{\partial A_k}{\partial t} \, dV$$
$$= \int -\sum_i J_i \cdot \nabla \frac{\partial \mu_i}{\partial t} - \sum_{i,k} v_{ik} v_k \frac{\partial \mu_i}{\partial t} \, dV,$$

and, therefore,

$$\frac{d_X P}{dt} = \int \sum_i \nabla \cdot J_i \frac{\partial \mu_i}{\partial t} - \sum_{i,k} v_{ik} v_k \frac{\partial \mu_i}{\partial t} \, dV$$
$$= \int \sum_{i,j} \frac{\partial \mu_i}{\partial c_j} \frac{\partial c_j}{\partial t} \nabla \cdot J_i - \sum_{i,j,k} v_{ik} v_k \frac{\partial \mu_i}{\partial c_j} \frac{\partial c_j}{\partial t} \, dV$$

under suitable boundary conditions. Then the second equation of (1.248) guarantees

$$\frac{d_X P}{dt} = -\int \sum_{i,j} \frac{\partial \mu_i}{\partial c_j} \frac{\partial c_i}{\partial t} \frac{\partial c_j}{\partial t} \, dV.$$

This quantity is non-negative around the stable equilibrium by (1.232), and thus, the stationary state near a linearly stable equilibrium is defined by the zero of this value.

Phenomenological Relation

Near from the equilibrium, the flux is proportional to the associated force; the diffusion flux is the product of the concentration gradient and the diffusion coefficient; the elecrtic current is that of the electro-static potential and the electric conductivity; and the heat flux is that of the thermal gradient and thermal conductivity, and so forth, referred to as Newton-Fourier-Fick's law or Ohm's law.
These relations are summarized as

$$J_i = L_i X_i,$$

or

$$J_i = \sum_j L_{ij} X_j,$$

regarding the effect of interaction, where L_{ij}'s are constant. Then Onsager's reciprocity relation is indicated by

$$L_{ij} = L_{ji}, \tag{1.251}$$

and (1.250) reduces to

$$\Phi = \sum_i J_i X_i = \sum_{i,j} L_{ij} X_i X_j \geq 0.$$

Here, the equality indicates the equilibrium again. Thus, we obtain

$$\begin{aligned}\sum_i X_i \frac{dJ_i}{dt} &= \sum_{i,j} L_{ij} X_i \frac{dX_j}{dt} \\ &= \sum_{i,j} L_{ij} X_j \frac{dX_i}{dt} = \sum_i J_i \frac{dX_i}{dt}\end{aligned}$$

and hence

$$\frac{d_X P}{dt} = \frac{d_J P}{dt} = \frac{1}{2}\frac{dP}{dt},$$

which is non-negative near a linearly stable equilibrium.

Principle of Detailed Balance

In the reaction system

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} B \tag{1.252}$$

the phenomenological coefficient L defined by $J_{ch} = LA$ is equal to

$$L = \frac{k_1 \overline{c_A}}{RT}, \tag{1.253}$$

where $\overline{c_A}$ denotes the equilibrium concentration of A.
In fact, this system is described by

$$\begin{aligned}\frac{dc_A}{dt} &= -k_1 c_A + k_{-1} c_B \\ \frac{dc_B}{dt} &= k_1 c_A - k_{-1} c_B,\end{aligned}$$

and the chemical reaction rate from A to B is equal to

$$J_{ch} = -\frac{dc_A}{dt} = \frac{dc_B}{dt}.$$

In the equilibrium, we have

$$k_1 \overline{c_A} = k_{-1} \overline{c_B}, \tag{1.254}$$

where $\overline{c_A}$ and $\overline{c_B}$ are equilibrium concentrations, regarded as local densities. Here, we define the equilibrium constant by

$$K = \frac{k_{-1}}{k_1} = \frac{\overline{c_A}}{\overline{c_B}}.$$

Fluctuations from the equilibrium of these concentrations are denoted by

$$\alpha_A = c_A - \overline{c_A}$$
$$\alpha_B = c_B - \overline{c_B},$$

where it is assumed that

$$\left|\frac{\alpha_A}{\overline{c_A}}\right|, \quad \left|\frac{\alpha_B}{\overline{c_B}}\right| \ll 1.$$

Using mass conservation

$$c_A + c_B = \overline{c_A} + \overline{c_B}$$

and (1.254), we obtain

$$\begin{aligned} J_{ch} &= k_1 c_A - k_{-1} c_B = k_1 \alpha_A - k_{-1} \alpha_B \\ &= \alpha_A (k_1 + k_{-1}) = k_1 \alpha_A (1 + K). \end{aligned}$$

Chemical affinity, on the other hand, is defined by

$$A = \mu_A - \mu_B$$

because $\nu_A = 1$ and $\nu_B = -1$ follow from (1.252).
Since $\overline{\mu_A} = \overline{\mu_B}$, we have

$$\mu_A^0 + RT \log \overline{c_A} = \mu_B^0 + RT \log \overline{c_B}$$

by (1.231). Then it holds that

$$\begin{aligned} A &= \mu_A - \mu_B \\ &= \mu_A^0 + RT \log \overline{c_A} + RT \log\left(1 + \frac{\alpha_A}{\overline{c_A}}\right) \\ &\quad - \mu_B^0 - RT \log \overline{c_B} - RT \log\left(1 + \frac{\alpha_B}{\overline{c_B}}\right) \\ &= RT\left[\log\left(1 + \frac{\alpha_A}{\overline{c_A}}\right) - \log\left(1 + \frac{\alpha_B}{\overline{c_B}}\right)\right] \\ &\sim RT\left(\frac{\alpha_A}{\overline{c_A}} - \frac{\alpha_B}{\overline{c_B}}\right) \\ &= RT \frac{\alpha_A}{\overline{c_A}}(1 + K). \end{aligned} \tag{1.255}$$

Thus, we obtain (1.253) by $J_{ch} = LA$.
If there is

$$B \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} C, \qquad C \underset{k_{-3}}{\overset{k_3}{\rightleftarrows}} A$$

besides (1.252), then we obtain three chemical reaction rates and three chemical affinities denoted by

$$
\begin{aligned}
J_1 &= k_1 c_A - k_{-1} c_B \\
J_2 &= k_2 c_B - k_{-2} c_C \\
J_3 &= k_3 c_C - k_{-3} c_A
\end{aligned}
\tag{1.256}
$$

and

$$
\begin{aligned}
A_1 &= \mu_A - \mu_B \\
A_2 &= \mu_B - \mu_C \\
A_3 &= \mu_C - \mu_A,
\end{aligned}
$$

respectively. It holds that

$$A_1 + A_2 + A_3 = 0$$

and the dissipation function is defined by

$$\Phi = J_1 A_1 + J_2 A_2 + J_3 A_3 = (J_1 - J_3) A_1 + (J_2 - J_3) A_2.$$

Thus, the phenomenological relation guarantees

$$
\begin{aligned}
J_1 - J_3 &= L_{11} A_1 + L_{12} A_2 \\
J_2 - J_3 &= L_{21} A_1 + L_{12} A_2.
\end{aligned}
\tag{1.257}
$$

In the equilibrium, we have $\overline{\mu_A} = \overline{\mu_B} = \overline{\mu_C} = 0$ and, therefore, $A_1 = A_2 = 0$ and $J_1 = J_2 = J_3$. However, principle of detailed balance requires

$$J_1 = J_2 = J_3 = 0 \tag{1.258}$$

in the equilibrium. This equality implies

$$
\begin{aligned}
k_1 \overline{c_A} &= k_{-1} \overline{c_B} \\
k_2 \overline{c_B} &= k_{-2} \overline{c_C} \\
k_3 \overline{c_C} &= k_{-3} \overline{c_A}.
\end{aligned}
$$

Using the perturbations α_i from the equilibrium $\overline{c_i}$, we obtain

$$
\begin{aligned}
J_1 &= k_1 \alpha_A - k_{-1} \alpha_B \\
J_2 &= k_2 \alpha_B - k_{-2} \alpha_C \\
J_3 &= k_3 \alpha_C - k_{-3} \alpha_A
\end{aligned}
\tag{1.259}
$$

by (1.256). Since (1.255) reads

$$A_1 = \frac{RT}{k_1 \overline{c_A}} (k_1 \alpha_A - k_{-1} \alpha_B),$$

it holds that

$$J_1 = \frac{k_1 \overline{c_A}}{RT} A_1,$$

by (1.255) and similarly

$$J_2 = \frac{k_2\overline{c_B}}{RT}A_2$$
$$J_3 = \frac{k_3\overline{c_C}}{RT}(A_1 + A_2).$$

It holds that

$$J_1 - J_3 = \frac{k_1\overline{c_A} + k_3\overline{c_C}}{RT}A_1 + \frac{k_3\overline{c_C}}{RT}A_2$$
$$J_2 - J_3 = \frac{k_3\overline{c_C}}{RT}A_1 + \frac{k_2\overline{c_B} + k_3\overline{c_C}}{RT}A_2, \tag{1.260}$$

and hence we obtain $L_{12} = L_{21}$ by comparing (1.257) and (1.260). Thus the principle of detailed balance (1.258) implies reciprocity, (1.251).

1.4.5. Summary

We have described mathematical modelling of phase transition, reviewing the theory of non-equilibrium thermodynamics.

(1) The Fix-Caginalp equation is a combination of the enthalpy equation of the two-phase Stefan problem and the model (A) equation using a modified Ginzburg-Landau's free energy.
(2) Since this system is not consistent with the theory of thermodynamics, the Penrose-Fife theory is adopted if the temperature is far-from-equilibrium. Then we obtain the Penrose-Fife equation and the coupled Cahn-Hilliard equation for the non-isothermal phase transition and phase separation, respectively.
(3) Theory of non-equilibrium thermodynamics is concerned with the open system provided with the chemical reaction. It formulates the balance laws of the entropy, the inner energy, and the material density per unit volume, using the local equilibirum principle. Consequently, the solution to this system has a different feature from that of model (A), model (B), and model (C) equations.
(4) The stationary state of this system of balance laws casts the driving force to the non-equilibrium from the equilibrium, but its control region is restricted to near-from-equilibrium because the variational structure is lost in far-from-equilibrium.

1.5. Phase Fields

Regarding the order parameter as the "field component," we can see the semi-dual variational structure in several phase field equations concerning critical phenomena. Particularly, the nonlinear eigenvalue problem with non-local term arises as the stationary state of the closed system. Illustrating the profile of the total set of stationary solutions and distinguishing their stability and instability are the first step to clarify the process of self-organization. This section concludes the first chapter by examining these variational structures sealed in several mathematical models of self-assembly.

1.5.1. Fix-Caginalp Equation

If the system is involved by the quantity other than the order parameter, several model (C) equations are formulated in phenomenology. This section is devoted to the mathematical study of several model (C) equations. Actually, some of them take the Lyapunov function, and then we obtain the semi-unfolding-minimality concerning the order parameter, which provides stability of the linearly stable order parameter. This property is the case of the Penrose-Fife equation for phase transition,[252] the coupled Cahn-Hilliard equation for phase separation,[7,252] and the Falk-Pawlow equation for shape memory alloys.[91,92,249,250] Although the well-posedness of these equations have been studied by several authors,[34,355] this variational structure provides a new point of view to the global dynamics of these equations.

Thermally Open System

Historically, the first model (C) equation was the Fix-Caginalp equation described in §1.4.1, concerning non-isothermal phase transition. If $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$, it is defined by

$$\begin{aligned} \tau\varphi_t &= \xi^2\Delta\varphi + (\varphi - \varphi^3) + 2u \\ u_t + \frac{\ell}{2}\varphi_t &= \kappa\Delta u \qquad \text{in } \Omega \times (0,T), \end{aligned} \tag{1.261}$$

where $\tau = K^{-1} > 0$, $\ell > 0$, $\kappa > 0$, $\varphi = \varphi(x,t)$, and $u = u(x,t)$ denote the relaxization time, the latent heat, the conductivity, the order parameter, and the relative temperature, respectively. It is a coupling of the model (A) equation using the free energy

$$\mathscr{F}_u(\varphi) = \int_\Omega \frac{\xi^2}{2}|\nabla\varphi|^2 + W(\varphi) - 2u\varphi \, dx$$

and the enthalpy equation of the two-phase Stefan problem

$$\left(u + \frac{\ell}{2}\varphi\right)_t = \kappa\Delta u,$$

where $W(\varphi) = \frac{1}{4}\left(\varphi^2 - 1\right)^2$. This free energy prescribes that the equilibrium is $\varphi = \pm 1$ with $u = 0$, while we can show the semi-duality in the stationary state. Let us recall that $(\, , \,)$ denotes the L^2-inner product.

First, if this system is thermally open, then it holds that

$$\frac{\partial\varphi}{\partial\nu} = u = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.262}$$

In this case, we have

$$\begin{aligned} &\tau\|\varphi_t\|_2^2 = -\frac{\xi^2}{2}\frac{d}{dt}\|\nabla\varphi\|_2^2 - \frac{d}{dt}\int_\Omega W(\varphi) + 2(u,\varphi_t) \\ &\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{\ell}{2}(\varphi_t, u) = -\kappa\|\nabla u\|_2^2 \end{aligned}$$

and, therefore,

$$\frac{d}{dt}\left\{\frac{1}{2}\|u\|_2^2+\frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2+\frac{\ell}{4}\int_\Omega W(\varphi)\right\} = -\frac{\tau\ell}{4}\|\varphi_t\|_2^2-\kappa\|\nabla u\|_2^2 \\ \leq 0. \tag{1.263}$$

Thus

$$L(u,\varphi)=\frac{1}{2}\|u\|_2^2+\frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2+\frac{\ell}{4}\int_\Omega W(\varphi) \tag{1.264}$$

acts as a Lyapunov function. The functional $L = L(u,\varphi)$ is not provided with the hook term

$$\int_\Omega u\varphi,$$

and the Fix-Caginalp equation is not a gradient system derived from $L(u,\varphi)$.
In the stationary state, we have

$$u=\overline{u}\equiv 0$$

by putting $\partial_t\cdot = 0$ in the enthalpy equation

$$\left(u+\frac{\ell}{2}\varphi\right)_t=\kappa\Delta u \quad \text{in } \Omega\times(0,T), \qquad u=0 \quad \text{on } \partial\Omega\times(0,T),$$

and, therefore, the stationary $\varphi = \overline{\varphi}$ is defined by

$$-\xi^2\Delta\varphi=\varphi-\varphi^3 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu}=0 \quad \text{on } \partial\Omega, \tag{1.265}$$

putting $\partial_t\cdot = 0$ in the order parameter equation.
This elliptic problem concerning the stationary order parameter has the variational structure defined by Ginzburg-Landau's free energy,

$$\mathscr{F}(\varphi)=\int_\Omega \frac{\xi^2}{2}|\nabla\varphi|^2+W(\varphi)\,dx, \qquad \varphi\in H^1(\Omega),$$

and, therefore, it is equivalent to $\delta\mathscr{F}(\varphi)=0$ for $\varphi\in H^1(\Omega)$. Then we can confirm the semi-unfolding-minimality

$$L(u,\varphi)\geq L(\overline{u},\varphi)=\frac{\ell}{4}\mathscr{F}(\varphi).$$

From the argument described in §§1.2.6 and 1.3.4, this property implies that $(\varphi,u)=(\overline{\varphi},\overline{u})$ is dynamically stable in φ component if φ is a linearly stable critical point of $\mathscr{F}(\varphi)$ defined for $\varphi\in H^1(\Omega)$.

Thermal Analysis

If this system is thermally closed, then it holds that

$$\frac{\partial\varphi}{\partial\nu}=\frac{\partial u}{\partial\nu}=0 \qquad \text{on } \partial\Omega\times(0,T) \tag{1.266}$$

instead of (1.262). Equality (1.263) is valid even in this case, and $L(u,\varphi)$ of (1.264) is again a Lyapunov function of this system. Total enthalpy, on the other hand, is preserved in this case, and it holds that

$$\frac{d}{dt}\int_\Omega u+\frac{\ell}{2}\varphi\, dx=0.$$

From the enthalpy equation

$$\left(u+\frac{\ell}{2}\varphi\right)_t=\kappa\Delta u \quad \text{in } \Omega\times(0,T), \qquad \frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega\times(0,T),$$

the stationary u state of this closed system is a constant. This unknown constant $u=\overline{u}$ is determined by the prescribed total enthalpy,

$$\int_\Omega u+\frac{\ell}{2}\varphi\, dx=a.$$

Thus, the stationary state is defined by

$$\begin{aligned}&-\xi^2\Delta\varphi=\varphi-\varphi^3+2\overline{u} \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu}=0 \quad \text{on } \partial\Omega\\ &\overline{u}\,|\Omega|+\frac{\ell}{2}\int_\Omega\varphi=a\end{aligned}$$

or, equivalently,

$$\begin{aligned}&-\xi^2\Delta\varphi=\varphi-\varphi^3+\frac{2}{|\Omega|}\left(a-\frac{\ell}{2}\int_\Omega\varphi\right) && \text{in } \Omega\\ &\frac{\partial\varphi}{\partial\nu}=0 && \text{on } \partial\Omega.\end{aligned} \tag{1.267}$$

Regarding this a as an eigenvalue, we see that this physically closed stationary state of Fix-Caginalp equation is realized as a nonlinear eigenvalue problem with non-local term similar to (1.25).

The problem (1.267) has a variational function

$$\mathscr{F}_a(\varphi)=\frac{\xi^2}{2}\|\nabla\varphi\|_2^2+\int_\Omega W(\varphi)+\frac{2}{\ell\,|\Omega|}\left(a-\frac{\ell}{2}\int_\Omega\varphi\right)^2 \tag{1.268}$$

defined for $\varphi\in H^1(\Omega)$, and it is equivalent to $\delta\mathscr{F}_a(\varphi)=0$, that is this $\varphi\in H^1(\Omega)$ satisfies

$$\left.\frac{d}{ds}\mathscr{F}_a(\varphi+s\psi)\right|_{s=0}=0$$

for any $\psi\in H^1(\Omega)$. Then the semi-unfolding is obtained by

$$\begin{aligned}L(\overline{u},\varphi)&=\frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2+\frac{\ell}{4}\int_\Omega W(\varphi)+\frac{1}{2}\overline{u}^2\,|\Omega|\\ &=\frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2+\frac{\ell}{4}\int_\Omega W(\varphi)+\frac{1}{2\,|\Omega|}\left(a-\frac{\ell}{2}\int_\Omega\varphi\right)^2\\ &=\frac{\ell}{4}\mathscr{F}_a(\varphi),\end{aligned}$$

using

$$\int_\Omega \overline{u} + \frac{\ell}{2}\varphi \; dx = a.$$

This property implies the semi-minimality

$$L(\overline{u}, \varphi) = \frac{\ell}{4}\mathscr{F}_a(\varphi) \leq L(u, \varphi),$$

by

$$\int_\Omega u + \frac{\ell}{2}\varphi \; dx = a$$
$$\left(\frac{1}{|\Omega|}\int_\Omega u\right)^2 \leq \frac{1}{|\Omega|}\int_\Omega u^2.$$

Thus, a linearly stable critical function $\overline{\varphi}$ of $\mathscr{F}_a$ is dynamically stable. More precisely, if the self-adjoint operator A_a in $L^2(\Omega)$ with the domain

$$D(A_a) = \left\{ \psi \in H^1(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \;\text{ on } \partial\Omega \right\} \tag{1.269}$$

defined by

$$A_a \psi = -\xi^2 \Delta\psi + \overline{\varphi}^3 \psi - \overline{\varphi}\psi - \frac{1}{|\Omega|}\int_\Omega (a - \frac{\ell}{2}\overline{\varphi})\psi$$

is positive, then each $\varepsilon > 0$ admits $\delta > 0$ such that

$$\begin{array}{l} L(u_0, \varphi_0) - L(\overline{u}, \overline{\varphi}) < \delta \\ a = \overline{u}\,|\Omega| + \frac{\ell}{2}\int_\Omega \overline{\varphi} = \int_\Omega u_0 + \frac{\ell}{2}\varphi_0 \; dx \end{array} \quad \Rightarrow \quad \sup_{t \geq 0} \|\varphi(\cdot, t) - \overline{\varphi}\|_{H^1} < \varepsilon.$$

Enthalpy Analysis

If we regard

$$L = \frac{1}{2}\|u\|_2^2 + \frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2 + \frac{\ell}{4}\int_\Omega W(\varphi)$$

as a functional of $H = u + \frac{\ell}{2}\varphi$ and φ, then the above structure of semi-duality is derived from the formulation of §1.3.4.

In fact, in this case we obtain the hook term with

$$\begin{aligned} L(H, \varphi) &= \frac{1}{2}\left\| H - \frac{\ell}{2}\varphi \right\|_2^2 + \frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2 + \frac{\ell}{4}\int_\Omega W(\varphi) \\ &= \frac{1}{2}\|H\|_2^2 - \frac{\ell}{2}(H, \varphi) + \frac{\ell^2}{8}\|\varphi\|_2^2 + \frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2 + \frac{\ell}{4}\int_\Omega W(\varphi). \end{aligned}$$

Moreover, since

$$
\begin{aligned}
L_\varphi &= -\frac{\ell}{2}H + \frac{\ell^2}{4}\varphi - \frac{\ell\xi^2}{4}\Delta\varphi + \frac{\ell}{4}\left(\varphi^3 - \varphi\right) \\
&= -\frac{\ell}{4}\left\{\xi^2\Delta\varphi + \varphi - \varphi^3 + 2u\right\} \\
L_H &= H - \frac{\ell}{2}\varphi = u,
\end{aligned}
$$

the materially closed Caginalp-Fix equation is the combination of model (A) and model (B) equations derived from this Lagrangian, that is

$$
\begin{aligned}
&\frac{\ell\alpha}{4}\varphi_t = -L_\varphi \\
&H_t = \kappa\nabla\cdot\nabla L_H \qquad \text{in } \Omega\times(0,T) \\
&\frac{\partial}{\partial\nu}L_H = \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\times(0,T).
\end{aligned}
$$

This Lagrangian takes the form

$$
\frac{2}{\ell}L(H,\varphi) = G(\varphi) + F^*(H) - \langle H,\varphi\rangle
$$

for

$$
\begin{aligned}
G(\varphi) &= \frac{\xi^2}{4}\|\nabla\varphi\|_2^2 + \frac{1}{2}\int_\Omega W(\varphi) + \frac{\ell}{4}\|\varphi\|_2^2 \\
F^*(H) &= \frac{1}{\ell}\|H\|_2^2, \qquad \int_\Omega H = a,
\end{aligned}
$$

and this F^* is regarded as a proper, convex, lower semi-continuous functional defined on $L^2(\Omega)$. These relations imply

$$
\inf_H \frac{2}{\ell}L(H,\varphi) = \frac{2}{\ell}L(\overline{H},\varphi) = J(\varphi),
$$

where

$$
\begin{aligned}
&J(\varphi) = G(\varphi) - F(\varphi) \\
&\varphi\in\partial F^*(\overline{H}) \;\Leftrightarrow\; \overline{H}\in\partial F(\varphi),
\end{aligned}
$$

and it holds that

$$
\begin{aligned}
F(\varphi) &= \sup_{H,\ \int_\Omega H = a}\left\{\langle\varphi,H\rangle - F^*(H)\right\} \\
&= \frac{\ell}{4}\left\|\varphi - \frac{1}{|\Omega|}\int_\Omega\varphi\right\|_2^2 - \frac{1}{\ell}\cdot\frac{a^2}{|\Omega|}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
J(\varphi) &= \frac{\xi^2}{4}\|\nabla\varphi\|_2^2 + \frac{1}{2}\int_\Omega W(\varphi) + \frac{\ell}{4}|\Omega|\left(\frac{1}{|\Omega|}\int_\Omega\varphi\right)^2 + \frac{1}{\ell}\cdot\frac{a^2}{|\Omega|} \\
&= \frac{1}{2}\mathscr{F}_a(\varphi),
\end{aligned}
$$

and, therefore,

$$\frac{\ell}{4}\mathscr{F}_a(\varphi) \le L(H,\varphi), \qquad \int_\Omega H = a.$$

These structures motivate us to examine the hyper-stability of the critical point, see §1.3.4. In fact, it holds that

$$\begin{aligned} L &= \frac{\ell}{4}\mathscr{F}_a(\varphi) + \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|\overline{u}\|_2^2 \\ &= \frac{\ell}{4}\mathscr{F}_a(\varphi) + \frac{1}{2}\|u-\overline{u}\|_2^2 \end{aligned}$$

for

$$\overline{u} = \frac{1}{|\Omega|}\left(a - \frac{\ell}{2}\int_\Omega \varphi\right) = \frac{1}{|\Omega|}\int_\Omega u,$$

and, therefore, the following theorem is obtained.[203,312]

Theorem 1.29. *If $\overline{\varphi}$ is a linearly stable critical point of $\mathscr{F}_a$ defined by (1.268) for $\varphi \in H^1(\Omega)$, then each $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\begin{aligned} & \|u_0 - \overline{u}\|_2 + \|\varphi_0 - \overline{\varphi}\|_{H^1} < \delta \\ & \int_\Omega u_0 + \frac{\ell}{2}\varphi_0 \, dx = a \\ & \Rightarrow \\ & \sup_{t\in[0,T)} \left\{ \|u(\cdot,t) - \overline{u}\|_2 + \|\varphi(\cdot,t) - \overline{\varphi}\|_{H^1} \right\} < \varepsilon, \end{aligned}$$

where $(u(\cdot,t),\varphi(\cdot,t)) \in C\left([0,T), H^1(\Omega)\times L^2(\Omega)\right)$ is a solution to (1.261)-(1.266),

$$\overline{u} = \frac{1}{|\Omega|}\left(a - \frac{\ell}{2}\int_\Omega \overline{\varphi}\right),$$

and $T > 0$ is arbitrary.

1.5.2. Penrose-Fife Equation

Penrose-Fife equation (1.247) is formulated by

$$\begin{aligned} & \theta_t + \lambda(\varphi)_t - \Delta\alpha(\theta) = 0 \\ & \varphi_t - \kappa\Delta\varphi + g(\varphi) - \alpha(\theta)\lambda'(\varphi) = 0 \quad \text{in } \Omega\times(0,T) \\ & \frac{\partial\alpha}{\partial\nu}(\theta) = \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega\times(0,T) \\ & \theta|_{t=0} = \theta_0(x) > 0 \qquad (1.270) \\ & \varphi|_{t=0} = \varphi_0(x) \quad \text{in } \Omega, \qquad (1.271) \end{aligned}$$

where $\Omega \subset \mathbf{R}^n$, $n = 1, 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$, φ and $\theta > 0$ are order parameter and absolute temperature, respectively,

$$\lambda(\varphi) = a_1\varphi^2 + a_2\varphi$$
$$g(\varphi) = \varphi^3 - \varphi$$
$$\alpha(\theta) = -\frac{1}{\theta},$$

$\kappa > 0$, $a_1 < 0$, and a_2 are constants, and $\lambda(\varphi)$ stands for the latent heat.
We note that this is a materially and thermodynamically closed system, and, consequently, there is a Lyapunov function, provided with the semi-unfolding-minimality. More precisely, first, the total energy is preserved,

$$\frac{d}{dt}\int_\Omega \theta + \lambda(\varphi)\, dx = 0,$$

and then

$$L(\theta,\varphi) = \frac{\kappa}{2}\|\nabla\varphi\|_2^2 - \int_\Omega \log\theta + \int_\Omega W(\varphi)$$

acts as the Lyapunov function:

$$\frac{d}{dt}L(\theta,\varphi) = -\|\varphi_t\|_2^2 - \|\nabla\alpha(\theta)\|_2^2 \le 0.$$

Again, this functional $L(\theta,\varphi)$ is not provided with the hook term, and (1.271) is not a gradient system derived from this functional.
In the stationary state, $\theta = \overline{\theta} > 0$ is a constant by (1.271), and $\varphi = \overline{\varphi}$ is a solution to

$$-\kappa\Delta\varphi + g(\varphi) + \frac{\lambda'(\varphi)}{\overline{\theta}} = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \tag{1.272}$$

satisfying

$$|\Omega|\,\overline{\theta} + \int_\Omega \lambda(\varphi) = a \tag{1.273}$$

for

$$a = \int_\Omega \theta_0 + \lambda(\varphi_0)\, dx. \tag{1.274}$$

Thus, it is formulated as a nonlinear eigenvalue problem with non-local term,

$$-\kappa\Delta\varphi + g(\varphi) + \frac{|\Omega|\,\lambda'(\varphi)}{a - \int_\Omega \lambda(\varphi)} = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega$$

comparable to (1.267).
This problem has a variational structure, and it is the Euler-Lagrange equation of

$$\mathscr{F}_a(\varphi) = \frac{\kappa}{2}\|\nabla\varphi\|_2^2 + \int_\Omega W(\varphi) - |\Omega|\log\left(\frac{a}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \lambda(\varphi)\right)$$

defined for $\varphi \in H^1(\Omega)$ satisfying

$$\int_\Omega \lambda(\varphi) < a,$$

where $W(\varphi)=\frac{1}{4}\left(\varphi^2-1\right)^2$. If

$$a=|\Omega|\overline{\theta}+\int_\Omega \lambda(\varphi),$$

then it holds that

$$\begin{aligned} L(\overline{\theta},\varphi) &= \frac{\kappa}{2}|\nabla\varphi|_2^2+\int_\Omega W(\varphi)-|\Omega|\log\left(\frac{a}{|\Omega|}-\frac{1}{|\Omega|}\int_\Omega \lambda(\varphi)\right) \\ &= \mathscr{F}_a(\varphi). \end{aligned}$$

This fact means the semi-unfolding. On the other hand, we obtain

$$L(\theta,\varphi)-L(\overline{\theta},\varphi)=-\int_\Omega \log\theta+|\Omega|\log\overline{\theta},$$

provided that

$$\int_\Omega \theta+\lambda(\varphi)\,dx=a,$$

and then the semi-minimality

$$L(\overline{\theta},\varphi)\leq L(\theta,\varphi)$$

follows from Jensen's inequality

$$\frac{1}{|\Omega|}\int_\Omega -\log\theta \geq -\log\left(\frac{1}{|\Omega|}\int_\Omega \theta\right).$$

To perform the rigorous analysis, we say that $[\theta,\varphi]:[0,T]\to L^2(\Omega)\times V$ is an H^1-solution to (1.271) if

$$\begin{aligned} & e=\theta+\lambda(\varphi)\in L^\infty(0,T;L^2(\Omega))\cap H^1(0,T;V^*) \\ & \varphi\in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;V)\cap H^1(0,T;L^2(\Omega)) \\ & \theta>0,\ \ \alpha=-\frac{1}{\theta}\in L^2(0,T;V) \end{aligned}$$

for $V=H^1(\Omega)$,

$$\begin{aligned} & \langle z,e_t\rangle_{V,V^*}+(\nabla z,\nabla\alpha)=0 \\ & (z,\varphi_t)+\kappa(\nabla z,\nabla\varphi)+\left(z,g(\varphi)-\alpha\lambda'(\varphi)\right)=0 \end{aligned}$$

a.e. $t\in(0,T)$ for each $z\in V$, and

$$\begin{aligned} & e|_{t=0}=\theta_0+\lambda(\varphi_0) \\ & \varphi|_{t=0}=\varphi_0. \end{aligned}$$

Comparison theorem and standard parabolic regularity are not valid to (1.271), and, therefore, deriving $\theta>0$ in the classical sense needs rather strong regularities of the solution. Here, we use the fact that the inverse function of $\alpha=\alpha(\theta)$ is realized as a maximum monotone graph

$$\rho(s)=\begin{cases} -\frac{1}{s}, & s<0 \\ +\infty, & s\geq 0 \end{cases}$$

in $\mathbf{R}\times\mathbf{R}$. Then realizing the relation $\alpha=-1/\theta$ in a function space, we obtain the following theorem.

Theorem 1.30 ([154]). *If*

$$\begin{aligned}
&0<\theta_0=\theta_0(x)\in L^2(\Omega)\\
&\log\theta_0\in L^1(\Omega)\\
&\alpha_0=-\frac{1}{\theta_0}\in V\\
&\varphi_0=\varphi_0(x)\in H^2(\Omega),
\end{aligned}$$

then there is a unique H^1 solution to (1.271) for any $T>0$.

Using the above mentioned well-posedness, we confirm the generation of a dynamical system, which is useful to classify the asymptotic profile of the solution.

Theorem 1.31 ([154]). *System (1.271) generates a weakly continuous dynamical system in the space*

$$X=\{(\theta,\varphi)\in L^2(\Omega)\times H^2(\Omega)\mid \frac{\partial\varphi}{\partial\nu}=0 \ \text{ on } \partial\Omega$$
$$\theta>0,\ \log\theta\in L^1(\Omega),\ \frac{1}{\theta}\in V\}.$$

We have

$$\begin{aligned}
&\alpha\in C([0,\infty);L^2(\Omega))\\
&\varphi\in C([0,\infty);V)\\
&\lim_{t\uparrow+\infty}\|\nabla\alpha(\cdot,t)\|_2=\lim_{t\uparrow+\infty}\|w_t(\cdot,t)\|_2=\lim_{t\uparrow+\infty}\|\theta_t(\cdot,t)\|_{V^*}=0\\
&\sup_{t\geq 0}\{\|\alpha(\cdot,t)\|_V+\|\varphi(\cdot,t)\|_{H^2}\}<+\infty
\end{aligned}$$

for $\alpha(\cdot,t)=\alpha(\theta)(\cdot,t)$, and, therefore, the orbit

$$\mathscr{O}=\{(\alpha(\cdot,t),\varphi(\cdot,t))\}_{t\geq 0},$$

described by (α,φ), is continuous and compact in $L^2(\Omega)\times V$, and in particular, it has a non-empty, connected, and compact ω-limt set $\omega(\alpha_0,\varphi_0)$ composed of stationary solutions.

More precisely, we define

$$\begin{aligned}
&\omega(\alpha_0,\varphi_0)=\{(\overline{\alpha},\overline{\varphi})\mid \text{there exists } t_k\uparrow+\infty \text{ such that}\\
&(\alpha(\cdot,t_k),\varphi(\cdot,t_k))\to(\overline{\alpha},\overline{\varphi})\quad \text{in } L^2(\Omega)\times V\}
\end{aligned}$$

and call $(\overline{\alpha},\overline{\varphi})\in L^2(\Omega)\times V$ a stationary solution if and only if $\overline{\theta}=-\frac{1}{\overline{\alpha}}>0$ is a constant and it holds that (1.272)-(1.274) for $\varphi=\overline{\varphi}$. The set of stationary states, denoted by E, is discrete if $n=1$, see.[190] For the other case, convergence to the stationary solution of $(\alpha(\cdot,t),\varphi(\cdot,t))$ in $L^2(\Omega)\times V$ is not obvious.

Here, the linearized stability of the stationary $\varphi = \overline{\varphi}$ is indicated by the positivity of the self-adjoint operator A_a in $L^2(\Omega)$ with the domain (1.269) defined by

$$A_a\psi = -\kappa\Delta\psi + \left(3\overline{\varphi}^3 - 1 + \frac{|\Omega|\,\lambda''(\overline{\varphi})}{a - \int_\Omega \lambda(\overline{\varphi})}\right)\psi + \frac{|\Omega|\int_\Omega \lambda'(\overline{\varphi})\psi}{(a - \int_\Omega \lambda(\overline{\varphi}))^2}.$$

In this case, we obtain its dynamical stability as well as the asymptotic stability.

Theorem 1.32 ([154]). *If $\overline{\varphi}$ is a linearly stable critical point of $\mathscr{F}_a$, and $\overline{\theta} > 0$ is a constant determined by (1.273) for $\varphi = \overline{\varphi}$, then each $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\begin{aligned} &\|\nabla(\varphi_0 - \overline{\varphi})\|_2 < \delta \\ &\left|\frac{1}{|\Omega|}\int_\Omega \log\theta_0 - \log\overline{\theta}\right| < \delta \\ &a = \int_\Omega (\theta_0 + \lambda(\varphi_0)) \end{aligned} \tag{1.275}$$

implies

$$\begin{aligned} &\sup_{t\geq 0}\|\nabla(\varphi(\cdot,t) - \overline{\varphi})\|_2 < \varepsilon \\ &\sup_{t\geq 0}\left|\frac{1}{|\Omega|}\int_\Omega \log\theta(\cdot,t) - \log\overline{\theta}\right| < \varepsilon. \end{aligned}$$

As is mentioned, each linearly stable critical point of $\mathscr{F}_a$ is isolated in V, and we have

$$(\alpha(\cdot,t), \varphi(\cdot,t)) \to (\overline{\alpha}, \overline{\varphi}) \qquad \text{in } L^2(\Omega)\times V$$

as $t \uparrow +\infty$ under the assumption of the above theorem, where $\overline{\alpha} = -\frac{1}{\overline{\theta}}$ with $\overline{\theta} > 0$. Classification of the stable stationary states, however, is not complete.

We can regard L as a functional of (e, φ), using the energy density $e = \theta + \lambda(\varphi)$,

$$L(e,\varphi) = \frac{\kappa}{2}\|\nabla\varphi\|_2^2 - \int_\Omega \log(e - \lambda(\varphi)) + \int_\Omega W(\varphi).$$

Then, (1.271) is formulated by a combination of model (A) - model (B) equations:

$$\begin{aligned} &\varphi_t = -L_\varphi \\ &e_t = \nabla\cdot\nabla L_e && \text{in } \Omega\times(0,T) \\ &\frac{\partial}{\partial\nu}L_e = \frac{\partial}{\partial\nu}\alpha = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned}$$

In this case, we obtain

$$\begin{aligned} &\inf_e L(e,\varphi) = L(\overline{e},\varphi) \\ &\overline{e} = \lambda(\varphi) - \frac{1}{|\Omega|}\int_\Omega \lambda(\varphi) + \frac{a}{|\Omega|} \\ &L(\overline{e},\varphi) = \mathscr{F}_a(\varphi), \end{aligned}$$

and this confirms the semi-unfolding-minimality

$$\mathscr{F}_a(\varphi) \le L(e,\varphi), \qquad \int_\Omega e = a$$

again. However, this L is not formulated as the convex-non-convex functional described in §1.3.4, and the hyper-stability to guarantee θ-stability is not certain to hold in spite of the second relation of (1.275).

1.5.3. Coupled Cahn-Hilliard Equation

The coupled Cahn-Hilliard equation describes non-isothermal phase separation using the absolute temperature $\theta = \theta(x,t) > 0$ and the order parameter $\varphi = \varphi(x,t)$ ([7,252]). In this case, we use the combination of model (B) - model (B) equations, and thus it holds that

$$\begin{aligned}
&\varphi_t = M_1 \Delta\left(-K_1\Delta\varphi + \frac{f_\varphi}{\theta}\right) \\
&e_t = -M_2\Delta\left(\frac{1}{\theta}\right) && \text{in } \Omega\times(0,T) \\
&\frac{\partial}{\partial\nu}\theta^{-1} = \frac{\partial\varphi}{\partial\nu} = 0 \\
&\frac{\partial}{\partial\nu}\left(-\kappa_1\Delta\varphi + \frac{f_\varphi}{\theta}\right) = 0 && \text{on } \partial\Omega\times(0,T),
\end{aligned} \tag{1.276}$$

where M_1, M_2, and K_1 are positive constants. In fact, Penrose-Fife's relations (1.239), (1.240), and (1.236) are summarized by

$$\begin{aligned}
&S(e,\varphi) = \int_\Omega s(e(x),\varphi(x)) - \frac{K_1}{2}|\nabla\varphi|^2 \, dx \\
&\frac{\partial s}{\partial\varphi} = -\frac{f_\varphi}{\theta} \\
&\frac{\partial s}{\partial e} = \frac{1}{\theta} \\
&e = f - \theta f_\theta,
\end{aligned}$$

and, therefore,

$$S(e,\varphi) = \int_\Omega \frac{K_1}{2}|\nabla\varphi|^2 + f_\theta \, dx.$$

Then, (1.276) is obtained by

$$\begin{aligned}
&\varphi_t = -M_1\Delta\frac{\delta S}{\delta\varphi} \\
&e_t = -M_2\Delta\frac{\delta S}{\delta e} && \text{in } \Omega\times(0,T) \\
&\frac{\partial}{\partial\nu}\frac{\delta S}{\delta\varphi} = \frac{\partial}{\partial\nu}\frac{\delta S}{\delta e} = 0 \\
&\frac{\partial\varphi}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T).
\end{aligned}$$

Thus this system is associated with two conservation laws and the entropy increasing.
In fact, first, (1.276) describes a kinetically and thermo-dynamically closed system, and we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega e = 0\\ \frac{d}{dt}\int_\Omega \varphi = 0.\end{aligned} \tag{1.277}$$

Next, we have

$$\begin{aligned}\int_\Omega \varphi_t\left(-K_1\Delta\varphi + \frac{f_\varphi}{\theta}\right) &= -M_1\int_\Omega \left|\nabla\left(-K_1\Delta\varphi + \frac{f_\varphi}{\theta}\right)\right|^2\\ &= \frac{K_1}{2}\frac{d}{dt}\|\nabla\varphi\|_2^2 + \int_\Omega \frac{\varphi_t f_\varphi}{\theta}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \frac{\varphi_t f_\varphi}{\theta} &= \int_\Omega \frac{f_t - \theta_t f_\theta}{\theta} = \int_\Omega \frac{e_t + \theta(f_\theta)_t}{\theta}\\ &= M_2\int_\Omega\left|\nabla\frac{1}{\theta}\right|^2 + \frac{d}{dt}\int_\Omega f_\theta,\end{aligned}$$

and hence it holds that

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{K_1}{2}\|\nabla\varphi\|_2^2 + \int_\Omega f_\theta\right\}\\ &= -M_1\int_\Omega\left|\nabla\left(-K_1\Delta\varphi + \frac{f_\varphi}{\theta}\right)\right|^2 - M_2\int_\Omega\left|\nabla\theta^{-1}\right|^2\\ &\le 0.\end{aligned} \tag{1.278}$$

Thus, the total heat energy and the total order parameter are preserved,

$$\begin{aligned}\int_\Omega f - \theta f_\theta\, dx = a\\ \int_\Omega \varphi = c,\end{aligned}$$

and the entropy acts as the Lyapunov function, re-formulated by

$$L(\theta,\varphi) = \frac{K_1}{2}\|\nabla\varphi\|_2^2 + \int_\Omega f_\theta.$$

The temperature is constant in the stationary state. It is denoted by $\theta = \overline{\theta} > 0$, and then it follows that

$$\begin{aligned}&-K_1\Delta\varphi + \frac{1}{\overline{\theta}}f_\varphi(\overline{\theta},\varphi) = \text{constant} \quad \text{in } \Omega\\ &\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\\ &\int_\Omega f(\overline{\theta},\varphi) - \overline{\theta} f_\theta(\overline{\theta},\varphi)\, dx = a\\ &\int_\Omega \varphi = c.\end{aligned} \tag{1.279}$$

Here, we use the free energy density of[252] described by

$$f(\theta,\varphi) = -c_v(\theta\log\theta+\delta) - \frac{\alpha}{2}\varphi^2 - \beta\varphi + \gamma - \theta s_0(\varphi)$$

with concave $s_0(\varphi)$, say,

$$s_0(\varphi) = -\frac{\varphi^4}{4},$$

where c_v, α, β, δ, and γ are positive constants. In this case, it holds that

$$f - \theta f_\theta = c_v\theta - \frac{\alpha}{2}\varphi^2 - \beta\varphi + \gamma - \delta c_v,$$

and the above stationary $\overline{\theta}$ is prescribed by

$$c_v\,|\Omega|\,\overline{\theta} = (\delta c_v - \gamma)\,|\Omega| + \int_\Omega \frac{\alpha}{2}\varphi^2 + \beta\varphi\,dx + a.$$

Thus writing

$$A(\varphi) = \frac{\alpha}{2}\varphi^2 + \beta\varphi + \delta c_v - \gamma,$$

we obtain

$$c_v\overline{\theta} = \frac{a}{|\Omega|} + \int_\Omega A(\varphi) \tag{1.280}$$

and

$$f(\theta,\varphi) = -c_v\theta\log\theta - A(\varphi) - \theta s_0(\theta) + \text{constant},$$

which guarantees

$$f_\varphi(\theta,\varphi) = -A'(\varphi) - \theta s_0'(\varphi). \tag{1.281}$$

Relations (1.279), (1.280), and (1.281) are summarized by

$$\begin{aligned}
&-K_1\Delta\varphi = \frac{c_v\,|\Omega|\,A'(\varphi)}{a+\int_\Omega A(\varphi)} + s_0'(\varphi) - \frac{c_v\int_\Omega A'(\varphi)}{a+\int_\Omega A(\varphi)} - \frac{1}{|\Omega|}\int_\Omega s_0'(\varphi) \quad \text{in } \Omega \\
&\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega, \qquad\qquad \int_\Omega \varphi = c
\end{aligned} \tag{1.282}$$

similarly to (1.272)-(1.273). Then, we see that (1.282) is the Euler-Lagrange equation of

$$\begin{aligned}
\mathscr{F}_a(\varphi) = {} & \frac{K_1}{2}\,\|\nabla\varphi\|_2^2 - c_v\,|\Omega|\log\left(a+\int_\Omega A(\varphi)\right) - \int_\Omega s_0(\varphi) \\
& - c_v\,|\Omega|\log(c_v\,|\Omega|)
\end{aligned}$$

defined for

$$\begin{aligned}
&\varphi \in H^1(\Omega), \qquad \int_\Omega \varphi = c \\
&a + \int_\Omega A(\varphi) > 0.
\end{aligned} \tag{1.283}$$

On the other hand, for the stationary $\theta = \overline{\theta}$ defined by (1.280), it holds that

$$L(\overline{\theta},\varphi) = \frac{K_1}{2}\|\nabla\varphi\|_2^2 + \int_\Omega f_\theta(\overline{\theta},\varphi)$$

with

$$\begin{aligned} f_\theta(\overline{\theta},\varphi) &= -c_v(\log\overline{\theta}+1) - s_0(\varphi) \\ c_v\,|\Omega|\log\overline{\theta} &= c_v\,|\Omega|\left\{\log\left(a+\int_\Omega A(\varphi)\right) - \log(c_v\,|\Omega|)\right\}. \end{aligned}$$

This property implies the semi-unfolding,

$$L(\overline{\theta},\varphi) = \mathscr{F}_a(\varphi).$$

Finally, we have

$$\begin{aligned} & f_\theta(\theta,\varphi) - f_\theta(\varphi,\overline{\theta}) = -c_v(\log\theta - \log\overline{\theta}) \\ & \frac{1}{|\Omega|}\int_\Omega \theta = \overline{\theta} \end{aligned}$$

by (1.280), while Jensen's inequality implies

$$\frac{1}{|\Omega|}\int_\Omega -\log\theta \geq -\frac{1}{|\Omega|}\int_\Omega \theta = -\log\overline{\theta}.$$

Thus, we obtain

$$\int_\Omega f_\theta(\theta,\varphi) - f_\theta(\overline{\theta},\varphi)\; dx \geq 0,$$

and hence the minimality

$$L(\theta,\varphi) \geq \mathscr{F}_a(\varphi) \tag{1.284}$$

for φ satisfying (1.283). These structures are sufficient to motivate the study of stationary solutions from the dynamical point of view. We obtain an analogous result to Theorem 1.32 if the friction term $\mu\Delta u_t$ is added to the right-hand side of the first equation to (1.276), see.[155]

1.5.4. Shape Memory Alloys

Hysteresis of the shape memory alloys is modelled by[91,92] for the one-dimensional case of $\Omega = (0,1)$, where u, ε, σ, e, q, h, and g denote the displacement, the strain, the stress, the inner energy, the heat flux, the load density, and the heat source density, respectively. More precisely, momentum and energy balances described by

$$\begin{aligned} & u_{tt} - \sigma_x + \mu_{xx} = h(x,t) \\ & e_t + q_x - \sigma\varepsilon_t - \mu\varepsilon_{xt} = g(x,t) \end{aligned}$$

are coupled with the structural relation

$$\varepsilon = u_x, \quad \sigma = \frac{\partial f}{\partial\varepsilon}, \quad \mu = \frac{\partial f}{\partial\varepsilon_x}, \quad e = f - \theta f_\theta, \tag{1.285}$$

where $f = f(\varepsilon, \varepsilon_x, \theta)$ denotes the free energy density defined by

$$
\begin{aligned}
f &= f_0(\theta) + \alpha_1(\theta - \theta_1) f_1(\varepsilon) + f_2(\varepsilon) + \frac{\gamma}{2}\varepsilon_x^2 \\
f_0(\theta) &= -c_v \theta \left(\log \frac{\theta}{\theta_2} - 1 \right) + \tilde{c} \\
f_1(\varepsilon) &= \varepsilon^2 \\
f_2(\varepsilon) &= -\alpha_2 \varepsilon^4 + \alpha_3 \varepsilon^6,
\end{aligned}
$$

and furthermore, Newton-Fourier-Fick's law is adopted:

$$
q = -\kappa \theta_x.
$$

Here, the thermodynamical relation

$$
\begin{aligned}
A &= U - TS \\
S &= -\left(\frac{\partial A}{\partial T} \right)_V
\end{aligned}
$$

in §1.4.2 is used in the last equation of (1.285), see (1.236). Then we obtain

$$
\begin{aligned}
& u_{tt} - \left(\frac{\partial f}{\partial \varepsilon} \right)_x + \gamma \partial_x^4 u = h \\
& e_t - \kappa \theta_{xx} - \left(\frac{\partial f}{\partial \varepsilon} \right) \varepsilon_t - \left(\frac{\partial f}{\partial \varepsilon_x} \right) \varepsilon_{xt} = g \quad \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} = \frac{\partial u_{xx}}{\partial \nu} = \frac{\partial \theta}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).
\end{aligned}
\tag{1.286}
$$

Since

$$
\begin{aligned}
\frac{\partial}{\partial t} f(\varepsilon, \varepsilon_x, \theta) &= \left(\frac{\partial f}{\partial \varepsilon} \right) \varepsilon_t + \left(\frac{\partial f}{\partial \varepsilon_x} \right) \varepsilon_{xt} + \left(\frac{\partial f}{\partial \theta} \right) \theta_t \\
e_t &= f_t - \theta_t f_\theta - \theta \frac{\partial}{\partial t} f_\theta,
\end{aligned}
$$

it holds that

$$
e_t - \left(\frac{\partial f}{\partial \varepsilon} \right) \varepsilon_t - \left(\frac{\partial f}{\partial \varepsilon_x} \right) \varepsilon_{xt} = -\theta \frac{\partial}{\partial t} f_\theta,
$$

and the second equation of (1.286) is described by

$$
\theta \frac{\partial}{\partial t} f_\theta + \kappa \theta_{xx} = g.
$$

This equation is equivalent to

$$
c_V \theta_t - \kappa \theta_{xx} - \alpha_1 \theta (\varepsilon^2)_t = g
$$

because

$$
f_\theta = f_0'(\theta) + \alpha_1 f_1(\varepsilon) = -c_V \log \frac{\theta}{\theta_2} + \alpha_1 \varepsilon^2,
$$

and, therefore, (1.286) means

$$
\begin{aligned}
& u_{tt} - \left(\frac{\partial f}{\partial \varepsilon}\right)_x + \gamma \partial_x^4 u = h \\
& c_v \theta_t - \kappa \theta_{xx} - \alpha_1 \theta \left(\varepsilon^2\right)_t = g \qquad \text{in } \Omega \times (0,T) \\
& \frac{\partial u}{\partial \nu} = \frac{\partial u_{xx}}{\partial \nu} = \frac{\partial \theta}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).
\end{aligned} \tag{1.287}
$$

See[34,351] for more on physical backgrounds and the well-posedness of (1.286).

Three-Dimensional Model

Three dimensional model[250] is concerned with the bounded domain $\Omega \subset \mathbf{R}^3$ with smooth boundary $\partial\Omega$. Thus $u = (u_i)_{i=1,2,3}$ and $\theta > 0$ again stand for the displacement and the absolute temperature, respectively. Then it is described by

$$
\begin{aligned}
& u_{tt} - \gamma Q u_t + \frac{\kappa}{4} QQu = \nabla \cdot F_{/\varepsilon}(\varepsilon,\theta) + h \\
& c(\varepsilon,\theta)\theta_t - \kappa \Delta \theta = \theta F_{/\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \gamma A \varepsilon_t : \varepsilon_t + g \\
& \qquad \text{in } \Omega \times (0,T)
\end{aligned} \tag{1.288}
$$

for

$$
Q(u) = \nabla \cdot A\varepsilon(u), \qquad A\varepsilon = \lambda(\mathrm{tr}\ \varepsilon)I + 2\mu\varepsilon,
$$

where λ and μ are the Lamé constants satisfying

$$
\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0, \tag{1.289}
$$

$\kappa > 0$ is the conductivity, and $\gamma \geq 0$ is the visicosity. Here, $\varepsilon = (\varepsilon_{ij})$ denotes the stress tensor defined by

$$
\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i/j} + u_{j/i}\right)
$$

for $\cdot/i = \frac{\partial}{\partial x_i}\cdot$, and

$$
\begin{aligned}
& \varepsilon_t = \varepsilon(u_t), \quad F_{/\varepsilon} = \left(\frac{\partial F}{\partial \varepsilon_{ij}}\right)_{ij}, \quad F_{/\theta\varepsilon} = \frac{\partial}{\partial\theta} F_{/\varepsilon} \\
& (a_{ij}) : (b_{ij}) = \sum_{i,j} a_{ij} b_{ij}.
\end{aligned}
$$

Furthermore, $F = F(\varepsilon,\theta)$ is the (elastic) free energy density and A is a fourth order linear elastic tensor determined by Hooke's law, and

$$
c(\varepsilon,\theta) = c_v - \theta F_{/\theta\theta}(\varepsilon,\theta)
$$

is the specific heat ratio. Thus (1.288) is a natural three-dimensional extension of (1.287).

We assume $h=0$ and $g=0$ in (1.288), and take the kinetically and thermodynamically closed system provided with the boundary condition

$$\begin{aligned}
&\mu\frac{\partial u}{\partial \nu}+(\lambda+\mu)\nu\nabla\cdot u=0\\
&\frac{\kappa}{4}\left\{\mu\frac{\partial}{\partial \nu}Qu+(\lambda+\mu)\nu\nabla\cdot Qu\right\}-\nu\cdot F_{/\varepsilon}=0\\
&\frac{\partial\theta}{\partial\nu}=0 \qquad\qquad \text{on } \partial\Omega\times(0,T).
\end{aligned} \tag{1.290}$$

In this case, we obtain the total momentum conservation

$$\frac{d}{dt}\int_\Omega \dot{u}=\int_\Omega u_{tt}=0. \tag{1.291}$$

Using

$$Aa:a=\left(\lambda+\frac{2\mu}{3}\right)(\mathrm{tr}\ a)^2+\frac{4\mu}{3}|a|^2\geq 0,$$

valid to any symmetric second order tensor $a=(a_{ij})$ with $a_{ji}=a_{ij}$, we obtain the entropy increasing

$$\frac{d}{dt}\int_\Omega c_\nu\log\theta-F_{/\theta}\ dx\geq 0. \tag{1.292}$$

In fact, the second equation of (1.288) is described by

$$c_\nu\theta_t-\kappa\Delta\theta=\theta\frac{\partial}{\partial t}F_{/\theta}+\gamma A\varepsilon_t:\varepsilon_t \tag{1.293}$$

and hence it holds that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega c_\nu\log\theta-F_{/\theta}\ dx&=\int_\Omega \kappa\theta^{-1}\Delta\theta+\gamma\theta^{-1}A\varepsilon_t:\varepsilon_t\ dx\\
&=\int_\Omega \kappa\theta^{-2}|\nabla\theta|^2+\gamma\theta^{-1}A\varepsilon_t:\varepsilon_t\ dx\\
&\geq 0.
\end{aligned}$$

This inequality means (1.292).
Since

$$Qu=\nabla\cdot(A\varepsilon(u)) \tag{1.294}$$

by the definition of A, we obtain the energy identity

$$\frac{d}{dt}\left\{\frac{1}{2}\|\dot{u}\|_2^2+\frac{\kappa}{8}\|Qu\|_2^2+\int_\Omega F-F_{/\theta}\theta+c_\nu\theta\ dx\right\}=0. \tag{1.295}$$

In fact, using the first equation of (1.288), we obtain

$$\begin{aligned}
\frac{d}{dt}\left\{\frac{1}{2}\|\dot{u}\|_2^2+\frac{\kappa}{8}\|Qu\|_2^2\right\}&=(u_{tt},u_t)+\frac{\kappa}{4}(Qu,Qu_t)\\
&=\gamma(Qu_t,u_t)-\int_\Omega F_{/\varepsilon}:\nabla u_t,
\end{aligned}$$

where (1.293) is applicable to the second term of the right-hand side. More precisely,

$$\begin{aligned}\int_\Omega F_{/\varepsilon} : \nabla u_t &= \int_\Omega F_{/\varepsilon} : \nabla \varepsilon_t = \frac{d}{dt}\int_\Omega F - \left(F_{/\theta}, \theta_t\right) \\ &= \frac{d}{dt}\int_\Omega F - F_{/\theta}\theta \; dx + \int_\Omega \theta \left(F_{/\theta}\right)_t \\ &= \frac{d}{dt}\int_\Omega F - F_{/\theta}\theta + c_v\theta \; dx - \gamma \int_\Omega A\varepsilon_t : \varepsilon_t,\end{aligned}$$

and, therefore,

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|\dot{u}\|_2^2 + \frac{\kappa}{8}\|Qu\|_2^2 + \int_\Omega F - F_{/\theta}\theta + c_v\theta \; dx\right\} \\ &\quad = \gamma\{(Qu_t, u_t) + (A\varepsilon_t, \varepsilon_t)\} \\ &\quad = \gamma\int_\Omega -A\varepsilon_t : \nabla u_t + A\varepsilon_t : \varepsilon_t \; dx \\ &\quad = \gamma\int_\Omega -A\varepsilon_t : \varepsilon_t + A\varepsilon_t : \varepsilon_t \; dx = 0\end{aligned}$$

by (1.289) and (1.294). This equality means (1.295).

The stationary state is described by the constant temperature $\theta = \overline{\theta} > 0$ and

$$\begin{aligned}&\frac{\kappa}{4} QQu = \nabla \cdot F_{/\varepsilon}(\varepsilon, \overline{\theta}) && \text{in } \Omega \\ &\mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu\nabla\cdot u = 0 \\ &\frac{\kappa}{4}\left\{\mu\frac{\partial}{\partial\nu}Qu + (\lambda+\mu)\nu\cdot\nabla Qu\right\} - \nu\cdot F_{/\varepsilon} = 0 && \text{on } \partial\Omega.\end{aligned} \tag{1.296}$$

Then

$$b = \frac{1}{2}\|\dot{u}\|_2^2 + \frac{\kappa}{8}\|Qu\|_2^2 + \int_\Omega F - F_{/\theta}\theta + c_v\theta \; dx \tag{1.297}$$

is preserved, and this b is determined by the initial data. Since $\dot{u} = 0$ in the stationary state, we obtain the coupling of (1.296) and

$$b = \frac{\kappa}{8}\|Qu\|_2^2 + \int_\Omega \left(F - F_{/\theta}\theta + c_v\theta\right)\big|_{\theta=\overline{\theta}} \tag{1.298}$$

to assign the stationary state.

The problem (1.296) has the variational functional

$$\mathscr{F}_{\overline{\theta}}(u) = \frac{\kappa}{8}\|Qu\|_2^2 + \int_\Omega F(\varepsilon, \overline{\theta})$$

defined for $u \in X = H^2(\Omega)^3$ because it is equivalent to

$$\frac{\kappa}{4}(Qu, Qv) + \int_\Omega F_{/\varepsilon}(\varepsilon, \overline{\theta}) : \nabla w = 0$$

for any $v \in H^2(\Omega)^3$ and $w = \frac{1}{2}(v + {}^t v)$. Then the second term is equal to

$$\int_\Omega F_{/\varepsilon}(\varepsilon, \overline{\theta}) : \nabla v,$$

and thus (1.296) is equivalent to $\delta \mathscr{F}_{\overline{\theta}}(u) = 0$.
Regarding

$$L(\theta, u) = \int_\Omega F_{/\theta} - c_v \log \theta \; dx$$

as a Lyapunov function, we can confirm the semi-unfolding. In fact, first, (1.298) implies

$$L(\overline{\theta}, u) = A(\varepsilon) - c_v |\Omega| \log \left(b - \mathscr{F}_{\overline{\theta}}(u) + \overline{\theta} A(\varepsilon) \right) + c_v |\Omega| \log (c_v |\Omega|) \tag{1.299}$$

for

$$A(\varepsilon) = \int_\Omega F_{/\theta}(\overline{\theta}, \varepsilon).$$

Next,

$$L_u(\overline{\theta}, u) = 0$$

means

$$\delta A(\varepsilon) - c_v |\Omega| \cdot \frac{-\delta \mathscr{F}_{\overline{\theta}}(u) + \overline{\theta} \delta A(\varepsilon)}{b - \mathscr{F}_{\overline{\theta}}(u) + \overline{\theta} A(\varepsilon)} = 0$$

or, equivalently,

$$\delta \mathscr{F}_{\overline{\theta}}(u) + \left(\frac{b - \mathscr{F}_{\overline{\theta}}(u) + \overline{\theta} A(\varepsilon)}{c_v |\Omega|} - \overline{\theta} \right) \delta A(\varepsilon) = 0.$$

Here, the second term of the right-hand side vanishes by (1.298), and, therefore, it holds that

$$L_u(\overline{\theta}, u) = 0 \quad \Leftrightarrow \quad \delta \mathscr{F}_{\overline{\theta}}(u) = 0.$$

To examine the semi-minimality, let $(\theta, u, \dot{u})$ be in (1.297), and $\overline{\theta} > 0$ be a constant satisfying (1.298). Then it holds that

$$c_v \int_\Omega \theta + \mathscr{F}_{\overline{\theta}}(u) + \int_\Omega F - F_{/\theta}\theta - F|_{\theta = \overline{\theta}} \; dx \leq b,$$

and, therefore, using

$$-\log \left(\frac{1}{|\Omega|} \int_\Omega \theta \right) \leq \frac{1}{|\Omega|} \int_\Omega (-\log \theta),$$

we obtain

$$\begin{aligned} L(\theta, u) &= \int_\Omega F_{/\theta} - c_v \log \theta \; dx \\ &\geq \int_\Omega F_{/\theta} - c_v |\Omega| \log \left(\frac{1}{|\Omega|} \int_\Omega \theta \right) \\ &\geq \int_\Omega F_{/\theta} - c_v |\Omega| \log \left(b - \mathscr{F}_{\overline{\theta}}(u) - \int_\Omega F - F_{/\theta}\theta - F|_{\theta = \overline{\theta}} \; dx \right) \\ &\qquad + c_v |\Omega| \log (c_v |\Omega|). \end{aligned}$$

Thus, it follows that

$$L(\theta,u) \geq L(\overline{\theta},u),$$

provided that

$$\int_\Omega F_{/\theta} \geq \int_\Omega F_{/\theta}\big|_{\theta=\overline{\theta}}$$
$$\int_\Omega F - F_{/\theta}\theta\, dx \geq \int_\Omega \left(F - F_{/\theta}\theta\right)\big|_{\theta=\overline{\theta}}. \tag{1.300}$$

Here, we note that inequalities (1.300) are valid if $F = F(\varepsilon,\theta)$ is linear in θ, as is defined by.[93] Only $F_{/\varepsilon}$ appears in system of equations (1.288), and, therefore, it suffices to exist $\tilde{F}$ linear in θ such that $\tilde{F}_{/\varepsilon} = F_{/\varepsilon}$.
Based on these considerations, we assign b by (1.297) using the initial value. In the case that $\overline{\theta} > 0$ is a constant and $\overline{u}$ is a linearly stable critical point of $\mathscr{F}_{\overline{\theta}} = \mathscr{F}_{\overline{\theta}}(u)$ satisfying (1.300) and (1.298) for $u = \overline{u}$, then the stationary state $(\theta,u,\dot{u}) = (\overline{\theta},\overline{u},0)$ is dynamically stable. It is formal if $\gamma = 0$ because these quantities are not sufficient to contol the well-posedness of this problem, see.[352,353]

One-Dimensional Case

In the case of one-space dimension, however, all the processes are justified. We reformulate the Falk model (1.287) as

$$\begin{aligned}
&u_{tt} + \gamma u_{xxxx} = (F_1'(u_x)\theta + F_2'(u_x))_x \\
&c_v\theta_t - \kappa\theta_{xx} = \theta F_1'(u_x)u_{tx} && \text{in } \Omega\times(0,T) \\
&u_x = u_{xxx} = \theta_x = 0 && \text{on } \partial\Omega\times(0,T) \\
&u\big|_{t=0} = u_0,\quad u_t\big|_{t=0} = u_1,\quad \theta\big|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned} \tag{1.301}$$

where $\Omega = (0,1)$ and F_1, F_2 are C^4-functions satisfying

$$F_1'(0) = 0$$
$$F_2 \geq -C.$$

If

$$(u_0,u_1,\theta_0) \in H^4(\Omega)\times H^2(\Omega)\times H^2(\Omega), \tag{1.302}$$

then there is a unique solution (u,θ) global in time such that

$$u \in C([0,+\infty),H^4(\Omega)),\quad u_t \in C([0,+\infty),H^2(\Omega))$$
$$\theta \in C([0,+\infty),H^2(\Omega)),\quad \theta_t \in L^2_{loc}(0,+\infty;H^1(\Omega)),$$

see.[313]
In the stationary state, θ is a constant denoted by $\overline{\theta}$, which is detemined by the energy conservation,

$$c_v\overline{\theta} + \frac{\gamma}{2}\|u_{xx}\|_2^2 + \int_\Omega F_2(u_x) = b,$$

where

$$b=\frac{1}{2}\|u_1\|_2^2+\frac{\gamma}{2}\|u_{0xx}\|_2^2+\int_\Omega F_2(u_{0x})+c_v\theta_0\,dx. \tag{1.303}$$

Thus the stationary state to $u=u(x)$ is formulated as the Euler-Lagrange equation

$$\delta J_b(u_x)=0,$$

where

$$J_b(u_x)=\frac{1}{c_v}\int_\Omega F_1(u_x)-\log\left(b-\frac{\gamma}{2}\|u_{xx}\|_2^2-\int_\Omega F_2(u_x)\right)$$
$$u_x\in H^1(\Omega). \tag{1.304}$$

Then the stationary solution $(\overline{u},\overline{\theta})$ to (1.301) is called linearly stable if $\overline{u}=\overline{u}(x)$ is a stable critical point of J_b, that is its second derivative is positive definite. From the momentum conservation, finally, we impose

$$\int_\Omega u_1=0 \tag{1.305}$$

for the stability of (u,θ) to hold because this equality is the case of the stationary state by $\overline{u}_t=0$.

Theorem 1.33 ([313]). *Let (1.302) hold in (1.301), and assume*

$$\inf_\Omega \theta_0\geq\theta_*$$

and (1.305), where $\theta_*>0$ is a constant. We take $b>0$ of (1.303) and suppose that $\overline{u}=\overline{u}(x)$ is a linearly stable critical point of $J_b=J_b(u_x)$ defined by (1.304). Let $\overline{\theta}>0$ be a constant satisfying

$$c_v\overline{\theta}+\frac{\gamma}{2}\|\overline{u}_{xx}\|_2^2+\int_\Omega F_2(\overline{u}_x)=b.$$

Then, this $(\overline{u},\overline{\theta})$ is dynamically stable in the sense that any $\varepsilon>0$ admits $\delta>0$ such that

$$\sup_{t\geq 0}\|(u(\cdot,t)-\overline{u})_x\|_{H^1(\Omega)}<\varepsilon$$
$$\sup_{t\geq 0}\left|\int_0^1\log\theta(\cdot,t)-\log\overline{\theta}\right|<\varepsilon$$

provided that

$$\sup_{t\geq 0}\|(u_0-\overline{u})_x\|_{H^1(\Omega)}<\delta$$
$$\left|\int_0^1\log\theta_0-\log\overline{\theta}\right|<\delta.$$

1.5.5. Time Map

Method of the time map is applicable to the stationary phase field models for the one-dimensional space domain.[270,271] This paragraph is devoted to the simplest application, (1.265) with $\Omega = (0,1)$. See[312] for (1.267).
First, in the general case of

$$\begin{aligned} &\xi^2 v'' + f(v) = 0, \qquad 0 < x < 1 \\ &v'(0) = v'(1) = 0 \end{aligned} \tag{1.306}$$

with $\xi > 0$, we take the solution $v = v(x,a)$ to

$$\begin{aligned} &\xi^2 v'' + f(v) = 0, \qquad 0 < x < 1, \\ &v'(0) = 0, \quad v(0) = a, \end{aligned}$$

assuming $f(a) > 0$ without loss of generarity. This assumption implies $v'(x) < 0$ for $0 < x \ll 1$, and we define the first zero of $v' = 0$, denoted by $x_0 \in (0,1]$, if it exists. Then, from the reflection argument we obtain the solution to (1.306) if and only if

$$x_0 = \frac{1}{k} \tag{1.307}$$

for some $k = 1, 2, \ldots$, standing for the mode of the solution.
Here, we have

$$\frac{1}{2}\xi^2 (v')^2 + F(v) = F(a)$$

for

$$F(s) = \int_0^s f(s')ds',$$

and, therefore, such x_0 exists if and only if there is $b(a)$ such that

$$b(a) < a, \qquad F(b(a)) = F(a), \qquad F|_{(b(a),a)} < F(a). \tag{1.308}$$

In this case, it holds that

$$\int_{v(x)}^a \frac{dv}{\sqrt{2\left(F(a) - F(v)\right)}} = \frac{x}{\xi}, \qquad 0 \le x \le x_0 \tag{1.309}$$

by

$$\xi \frac{dv}{dx} = -\sqrt{2\left(F(a) - F(v)\right)}, \qquad 0 \le x \le x_0,$$

and (1.307) is equivalent to

$$T(a) = \frac{1}{\xi k}, \tag{1.310}$$

where

$$T(a) = \int_{b(a)}^a \frac{dv}{\sqrt{2\left(F(v) - F(a)\right)}},$$

called the *time map*.
Since $f(v) = v - v^3$ in (1.265), we obtain

$$D(T) \equiv \{a \mid f(a) > 0,\ b(a) > -\infty\} = (0,1)$$
$$b(a) = -a,$$

and (1.309)-(1.310) are reduced to

$$\int_{-1}^{1} \frac{dy}{\sqrt{(1-y^2)(1-\ell^2 y^2)}} = \frac{1}{\xi k} \cdot \sqrt{1-\frac{a^2}{2}}$$
$$\int_{v(x)/a}^{1} \frac{dy}{\sqrt{(1-y^2)(1-\ell^2 y^2)}} = \frac{x}{\xi} \cdot \sqrt{1-\frac{a^2}{2}}$$

for $\ell = \frac{a}{\sqrt{2-a^2}}$. Using Jacobi's elliptic function $y = sn(w,\ell)$ defined by

$$w = \int_{y}^{1} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}},$$

thus we obtain the k-mode solution to (1.265) by

$$sn\left(\frac{1}{\xi k} \cdot \sqrt{1-\frac{a^2}{2}}, \frac{a}{\sqrt{2-a^2}}\right) = -1$$
$$v(x) = a \cdot sn\left(\frac{x}{\xi} \cdot \sqrt{1-\frac{a^2}{2}}, \frac{a}{\sqrt{2-\frac{a^2}{2}}}\right), \qquad 0 \le x \le 1/k,$$

where $0 < a < 1$. Then we can perform the asymptotic analysis as $a \downarrow 0$ and $a \uparrow 1$ to illustrate the solution set, see.[190]

1.5.6. Summary

Several phenomenological equations are provided with the Lyapunov function, and this functional induces a semi-dual variational structure to the stationary state, especially to the field component. The particle component, on the other hand, is sometimes trivial in the stationary state, which guarantees the dynamical stability of linearly stable stationary states. If the system is materially closed, then this stationary state is realized as a non-linear eigenvalue problem with non-local term. In some cases, we obtain the hyper-linear stability of the field component, which guarantees the stability of the particle component simultaneously.

(1) The structure of semi-dual variation is observed, especially in materially closed systems of phenomenology.
(2) Among them are the Caginalp-Fix equation, the Penrose-Fife equation, the coupled Cahn-Hilliard equation, and the Falk-Pawlow equation and the stationary state is described by the nonlinear eigenvalue problem with non-local term.

(3) Consequently, a stationary field state has a variational functional associated with a non-local term, and a linearly stable stationary state is dynamically stable.
(4) Besides the calculus of variation, the method of time map is applicable to the stationary state of these equations if the domain is one-dimensional.

Chapter 2

Scaling - Revealing Hierarchy

Details of the self-organization are hierarchical achievements between self-assembly and dissipative structures.[345] The latter is top-down and occurs in a far-from-equilibrium formulation of an open system that involves dissipation of entropy and is characterized by periodic structures, sprial waves, travelling waves, self-similar evolutions, and so forth.[227] The former is, as described in the previous chapter, bottom-up and controlled by the stationary state of the closed system which is involved by condensates, collapses, spikes, quantizations, free energy transmissions, variational structures, and so forth.
The control of the total set of stationary states to the global dynamics, however, is not restricted to thermodynamics. This profile is observed widely in mathematical models involved by the mean field hierarchy and sometimes referred to as the nonlinear spectral mechanics.[304] In more precise terms, there is a unified mathematical principle in each mean field hierarchy provided with the underlying physical principle, such as the conservation laws, decrease of the free energy, and so forth.
This chapter describes the *quantized blowup mechanism* which is one of the leading principle of self-assembly. It arises in self-interacting fluid, turbulence in the context of the *propagation of chaos*, mean field hierarchy derived from the *friction-fluctuation* self-interaction in the molecular kinetics, and gauge field concerning *condensate* of microscopic states. Actually, this profile of quantization is revealed by a blowup analysis which is one of the important products of the method of scaling and is valid even to the higher-space dimension.

2.1. Self-Interacting Continuum

The macroscopic state of particles that constitute the self-interacting fluid is formulated by the system of equations provided with the self-duality between the particle density and the field distribution. Here, the formation of the field is physical and is associated with the Poisson type equation. The stationary state is then described by the nonlinear eigenvalue problem with a non-local term because of the mass conservation, and this problem is provided with the variational structure from the energy conservation. Then the method of scaling detects the critical exponents for mass quantization. This section is devoted to the variational and scaling properties of the systems of self-interacting fluid such as the Euler-Poisson equation and the plasma confinement problem.

T. Suzuki, *Mean Field Theories and Dual Variation*, Atlantis Studies in Mathematics for Engineering and Science 2, DOI 10.1007/978-94-91216-22-0_2, © 2008 Atlantis Press/World Scientific

2.1.1. Self-Gravitating Fluids

Several fundamental equations of self-interacting fluids are provided with the semi-unfolding-minimality.

Euler-Poisson Equation

With the prescribed velocity $v = v(x) \in \mathbf{R}^3$, $x \in \mathbf{R}^3$ of the particle, we can define the local flow $\{T_t\}$ by (1.2). Thus $x(t) = T_t x_0$ indicates the solution to

$$\frac{dx}{dt} = v(x), \qquad x|_{t=0} = x_0.$$

If $f(x)$ stands for the observer of particles, then

$$u(x,t) = f(T_{-t}x)$$

indicates the distribution of particles detected by it at the time $-t$. From the semi-group property, it follows that

$$u(T_t y, t) = f(T_{-t} T_t y) = f(y)$$

for any $y \in \mathbf{R}^3$, and, therefore,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} u(T_t y, t) = \nabla u(T_t y, t) \cdot \frac{\partial}{\partial t} T_t y + u_t(T_t y, t) \\ &= v(T_t y) \cdot \nabla u(T_t y, t) + u_t(T_t y, t), \end{aligned}$$

or

$$\frac{\partial u}{\partial t} + v \cdot \nabla u = 0 \qquad \text{in } \mathbf{R}^3 \times (0,T).$$

Thus we obtain

$$\frac{Du}{Dt} = 0$$

for $u(x,t) = f(T_{-t}x)$.
We call

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla,$$

the *material derivative*, which indicates the differention in t of the transported quantity subject to v, and, therefore, the acceleration vector of this fluid is defined by

$$\frac{Dv}{Dt}.$$

From this observation, we obtain the *Euler equation* of motion,

$$\rho \frac{Dv}{Dt} = \rho F - \nabla p, \tag{2.1}$$

where ρ, p, and F denote the density, the pressure, and the outer force, respectively, and also the equation of continuity

$$\rho_t + \nabla \cdot (\rho v) = 0 \tag{2.2}$$

because $j = \rho v$ is nothing but the flux of ρ.
The self-gravitating fluid is associated with

$$F = \nabla\Gamma * \rho$$

using the Newton potential

$$\Gamma(x) = \frac{1}{4\pi|x|}.$$

These relations are combined with the state equation, and thus it holds that

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho v) = 0 \\ &\rho\left(v_t + (v\cdot\nabla)v\right) + \nabla p + \rho\nabla\Phi = 0 \\ &\Delta\Phi = \rho, \quad p = A\rho^\gamma \qquad\qquad \text{in } \mathbf{R}^3\times(0,T) \end{aligned} \tag{2.3}$$

using $\Phi = -F$, where $A > 0$ is a constant determined by the entropy, and $1 < \gamma < 2$ is the adiabatic constant. Here, we have $\gamma = 5/3, 7/5, \dots$ when the gas is mono-atomic, bi-atomic, and so on, and $\gamma = 4/3$ in the case of the excellent radiational pressure.[46]

Conservation Laws

From the first equation of (2.3), we obtain the non-negativity $\rho \geq 0$ from that of the initial value, $\rho|_{t=0} \geq 0$. The total mass conservation,

$$M = \int_{\mathbf{R}^3} \rho,$$

follows also from this equation, while the total energy conservation

$$E = \int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\, dx - \frac{1}{8\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{\rho(x)\rho(x)}{|x-x'|} dxdx' \tag{2.4}$$

is derived as follows.
First, we replace the second equation of (2.3) by

$$(\rho v)_t + \nabla\cdot(\rho v\otimes v) + \nabla p + \rho\nabla\Phi = 0, \tag{2.5}$$

using the first equation, where

$$v\otimes v = (v^i v^j) \qquad \text{for} \qquad v = (v^j)_{1\leq j\leq 3}.$$

Here, it holds that

$$\begin{aligned} \int_{\mathbf{R}^3} [\nabla\cdot(\rho v\otimes v)]\cdot v &= \int_{\mathbf{R}^3} \left[\partial_j(\rho v^i v^j)\right] v^i = -\int_{\mathbf{R}^3} \rho v^j v^i \partial_j v^i \\ &= -\frac{1}{2}\int_{\mathbf{R}^3} \rho v^j \partial_j |v|^2 = \frac{1}{2}\int_{\mathbf{R}^3} |v|^2 \nabla\cdot(\rho v) \\ &= -\frac{1}{2}\int_{\mathbf{R}^3} |v|^2 \rho_t \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^3}(\rho v)_t\cdot v &= \int_{\mathbf{R}^3}\rho_t\,|v|^2+\rho v_t\cdot v\,dx\\ &= \int_{\mathbf{R}^3}\rho_t\,|v|^2+\frac{1}{2}\rho\partial_t\,|v|^2\;dx,\end{aligned}$$

and, therefore,

$$\begin{aligned}\int_{\mathbf{R}^3}[(\rho v)_t+\nabla\cdot(\rho v\otimes v)]\cdot v &= \frac{1}{2}\int_{\mathbf{R}^3}|v|^2\rho_t+\rho\partial_t\,|v|^2\;dx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3}|v|^2\rho. \qquad (2.6)\end{aligned}$$

Next, we have

$$\begin{aligned}\int_{\mathbf{R}^3}\nabla p\cdot v = A\gamma\int_{\mathbf{R}^3}\rho^{\gamma-1}\nabla\rho\cdot v &= \frac{A\gamma}{\gamma-1}\int_{\mathbf{R}^3}\nabla\rho^{\gamma-1}\cdot\rho v\\ &= -\frac{A\gamma}{\gamma-1}\int_{\mathbf{R}^3}\rho^{\gamma-1}\nabla\cdot(\rho v) = \frac{A\gamma}{\gamma-1}\int_{\mathbf{R}^3}\rho^{\gamma-1}\rho_t\\ &= \frac{d}{dt}\int_{\mathbf{R}^3}\frac{p}{\gamma-1} \qquad (2.7)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^3}\rho\nabla\Phi\cdot v = -\int_{\mathbf{R}^3}\Phi\nabla\cdot(v\rho) &= \int_{\mathbf{R}^3}\Phi\rho_t\\ &= -\frac{1}{8\pi}\frac{d}{dt}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\rho(x,t)\rho(x',t)}{|x-x'|}dxdx'. \qquad (2.8)\end{aligned}$$

Thus (2.4) follows from (2.5)-(2.8).

Makino's Solution

Putting $w=\frac{2(A\gamma)^{1/2}}{\gamma-1}\cdot\rho^{(\gamma-1)/2}$, we can replace (2.3) by

$$\begin{aligned}&w_t+(v\cdot\nabla)w+\frac{\gamma-1}{2}w(\nabla\cdot v)=0\\ &v_t+(v\cdot\nabla)v+\frac{\gamma-1}{2}(w\cdot\nabla)w+\nabla\Phi=0\\ &\Delta\Phi=w^{2/(\gamma-1)} \qquad \text{in } \mathbf{R}^3\times(0,T), \qquad (2.9)\end{aligned}$$

where

$$\partial_t\nabla\Phi=\partial_t\nabla\Delta^{-1}\rho=\nabla\Delta^{-1}\rho_t=-\nabla\Delta^{-1}\nabla\cdot(\rho v)$$

is applicable.[193,196] More precisely, the Calderón-Zygmund operator, denoted by Op, is defined by the symbol

$$\left(\frac{\xi_i\xi_j}{|\xi|^2}\right)_{1\le i,j\le 3},$$

and then we obtain

$$\nabla\Phi = \nabla\Phi_0 + \int_0^t \left(\frac{\gamma-1}{2(A\gamma)^{1/2}}\right)^{2/(\gamma-1)} Op\left[w^{2/(\gamma-1)}v\right](\cdot,s)ds.$$

Using this formula, (2.9) is formulated as a quasilinear symmetric hyperbolic system, and then the general theory[162] guarantees the unique existence of the solution locally in time.[23,112] Particularly, based on Hardy-Littlewood's maximal function, we can construct the semi-group solution satisfying

$$(\rho(\cdot,t)^{(\gamma-1)/2}, v(\cdot,t)) \in C\left([0,T), H^s(\mathbf{R}^3)^4\right),$$

where $s > 5/2$ and $1 < \gamma < 9/5$. This property means the well-posedness of (2.9) in this space locally in time, and the existence time $T > 0$ is estimated from below by

$$\left\|\left(\rho_0^{(\gamma-1)/2}, v_0\right)\right\|_{H^s(\mathbf{R}^3)^4}.$$

The space $H^s(\mathbf{R}^3)^4$ with $s > 5/2$ contains the equilibrium, to be defined in the next paragraph, in the case of $\gamma = 6/5$. The assumption $s > 5/2$, on the other hand, is optimal in this approach of quasilinear symmetric hyperbolic system, in the sense that it comes from the fact that $H^\ell(\mathbf{R}^3)$ is an algebra if $\ell > 3/2$ and $\ell \notin \mathbf{N}$. However, if

$$(w,v) \in C^1\left([0,T)\times\mathbf{R}^3\right)^4$$

is a non-trivial radially symmetric solution to (2.9) with the support $\rho(\cdot,t)$ compact for each $t \in [0,T)$, then we obtain $T < +\infty$, using the Lagrangian coordinate.[195]

Second Moment

Weak formulation to (2.3) is described by

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^3} \rho\varphi = \int_{\mathbf{R}^3} \rho v\cdot\nabla\varphi \\
&\frac{d}{dt}\int_{\mathbf{R}^3} \rho v\cdot\psi = \int_{\mathbf{R}^3} \rho v\otimes v\cdot\nabla\psi + p\nabla\cdot\psi\, dx \\
&+\frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \left(\psi(x)-\psi(x')\right)\cdot\nabla\Gamma(x-x')\rho(x,t)\rho(x',t)dxdx'
\end{aligned}$$

for $\varphi \in C_0^\infty(\mathbf{R}^3)$ and $\psi \in C_0^\infty(\mathbf{R}^3)^3$, and hence it follows that

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{\mathbf{R}^3} \rho\varphi = &\int_{\mathbf{R}^3} \rho v\otimes v\cdot\nabla^2\varphi + p\Delta\varphi\, dx \\
&+\frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \left[\nabla\varphi(x)-\nabla\varphi(x')\right] \\
&\quad\cdot\nabla\Gamma(x-x')\rho(x,t)\rho(x',t)dxdx'.
\end{aligned} \tag{2.10}$$

Putting $\varphi = |x|^2$ in (2.10) is justified, for example, if $\rho = \rho(x,t)$ and $\rho v = (\rho v)(x,t)$ have compact support and are continuous in $x \in \mathbf{R}^3$ and continuously differentialble in $t \in (0,T)$.

In this case, it follows that

$$\frac{d^2}{dt^2}\int_{\mathbf{R}^3}\rho\,|x|^2 = \int_{\mathbf{R}^3} 2\rho\,|v|^2 + 6p\,dx - \frac{1}{4\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\rho(x,t)\rho(x',t)}{|x-x'|}dxdx'$$
$$\geq 2E + 2\left(3-\frac{1}{\gamma-1}\right)\int_{\mathbf{R}^3} p. \tag{2.11}$$

In particular, if $E > 0$ and $4/3 < \gamma < 2$, then its support radius must go to $+\infty$ as $t \uparrow +\infty$, see.[195,197,286]

2.1.2. Equilibrium

The classical equilibrium of (2.3) is defined by putting $v = 0$ and $\partial_t \cdot = 0$. This formulation means

$$\frac{A\gamma}{\gamma-1}\nabla\rho^{\gamma-1} + \nabla\Phi = 0 \qquad \text{in } \{\rho > 0\},$$

and, therefore,

$$\frac{A\gamma}{\gamma-1}\rho^{\gamma-1} - \Gamma*\rho = \text{constant in each component of } \{\rho > 0\}$$
$$\rho \geq 0 \quad \text{in } \mathbf{R}^3, \qquad \int_{\mathbf{R}^3}\rho = M \tag{2.12}$$

Putting

$$q = \frac{1}{\gamma-1}$$
$$u = \left(\frac{\gamma-1}{A\gamma}\right)^{(\gamma-1)/(\gamma-2)}\rho^{\gamma-1}$$
$$\Omega = \{u > 0\}$$

thus we obtain

$$-\Delta u = u^q,\ \ u > 0 \qquad \text{in } \Omega_i$$
$$u = 0,\ \ \Gamma * u^q = \text{constant} \qquad \text{on } \partial\Omega_i$$
$$\int_{\mathbf{R}^3} u^q = \left(\frac{A\gamma}{\gamma-1}\right)^{1/(\gamma-2)} M, \tag{2.13}$$

where Ω_i $(i = 1,2,\ldots)$ are the connected components of Ω.
Conversely, if (2.13) holds, then $u - \Gamma * u^q$ is harmonic in Ω_i and is a constant on $\partial\Omega_i$. This property implies the first equation of (2.13), and in this sense, (2.13) is equivalent to (2.12).

Radially Symmetric Equilibrium

If

$$-\Delta u = u^q,\ \ u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B \tag{2.14}$$

is satisfied for $B = B(0,R)$, then $u = u(|x|)$ follows from the general theory of Gidas-Ni-Nirenberg,[114] and, therefore, the first two lines of (2.13) are valid to $\Omega = \Omega_i = B$. It is known, on the other hand, that (2.14) has a solution if and only if $1 < q < 5$, that is $6/5 < \gamma < 2$, and, then, the solution denoted by

$$u = u_R(x),$$

is unique.[222]

We have the self-similarity in (2.14). If $u = u(x)$ solves

$$-\Delta u = u^q,$$

then so does $u_\mu = u_\mu(x)$ for $\mu > 0$, where

$$u_\mu(x) = \mu^{2/(q-1)} u(\mu x).$$

Thus it holds that

$$u_R(x) = R^{-2/(q-1)} U(R^{-1}x) \tag{2.15}$$

for the unique solution $U = U(y)$ to (2.14) with $B = B(0,1)$, that is $U = u_1$.

We have

$$\int_{B(0,R)} u_R^q(x)dx = R^{3-\frac{2q}{q-1}} \int_{B(0,1)} U(y)^q dy,$$

and, therefore, if $q \neq 3$ and $1 < q < 5$, then any $M > 0$ admits $R > 0$ such that (2.13) has a unique solution for $\Omega = B(0,R)$. Taking zero extension outside Ω, we obtain the equilibrium $(\rho, v) = (\overline{\rho}, 0)$ to (2.3) from this $u = u(x) \geq 0$ and

$$\overline{\rho} = \left(\frac{A\gamma}{\gamma - 1} \right)^{1/(2-\gamma)} u^q. \tag{2.16}$$

Therefore, if $\gamma \neq 4/3$ and $6/5 < \gamma < 2$, then (2.3) admits an equilibrium $(\rho, v) = (\overline{\rho}, 0)$ for each prescribed total mass $M > 0$. This $\rho = \rho(x) \geq 0$ is radially symmetric and has a compact support. Also, its total energy is defined by

$$\begin{aligned} E &= \int_{\mathbf{R}^3} \frac{A\overline{\rho}^\gamma}{\gamma - 1} - \frac{1}{8\pi} \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\overline{\rho}(x)\overline{\rho}(x')}{|x - x'|} dxdx' \\ &= aR^{-\frac{4}{q-1}+1} \int_{\mathbf{R}^3} U(y)^{q+1} dy - bR^{-\frac{4}{q-1}+1} \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{U(y)^q U(y')^q}{|y - y'|} dydy' \end{aligned}$$

for

$$\begin{aligned} a &= \frac{Ac^\gamma}{\gamma - 1} \\ b &= \frac{c^2}{8\pi} \\ c &= \left(\frac{A\gamma}{\gamma - 1} \right)^{\frac{1}{\gamma - 2}}. \end{aligned}$$

However,

$$\overline{w} = \overline{\rho}^{(\gamma-1)/2} \approx u^{1/2}$$

does not belong to $H^1(\mathbf{R}^3)$ because this u extended 0 outside $B = B(0,R)$ has a derivative gap on ∂B by the Hopf lemma.
In the case of $\gamma = 4/3$, on the other hand, there is $\lambda_* > 0$ such that if $\lambda = \lambda^*$ then the problem (2.13) admits a unique solution for $\Omega = B(0,R)$ with $R > 0$ arbitrary, and in the other case of $\lambda \neq \lambda^*$, there is no radially symmetric solution to (2.13), where

$$\lambda = \left(\frac{A\gamma}{\gamma-1}\right)^{1/(\gamma-2)} M.$$

Again, the equilibrium defined by this solution is not consistent to the function space described in the previous paragraph for the well-posedness of the Cauchy problem, $H^s(\mathbf{R}^3)$, $s > 5/2$ for $(\rho^{(\gamma-1)/2}, v)$. In fact, there is a derivative gap of $\overline{w} = \overline{\rho}^{(\gamma-1)/2}$ on the interface.
If $q \geq 5$, then

$$-\Delta v = v^q, \;\; v > 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{2.17}$$

admits no solution if Ω is star-shaped.[256] Radially symmetric solution, on the other hand, exists for any $q > 1$ if $\Omega = \{c < |x| < d\}$ ([167]). This solution, however, does not satisfy (2.13) because the values of $\Gamma * u^q$ on $|x| = c$ and $|x| = d$ are different.
While

$$-\Delta u = u^q, \;\; u > 0 \qquad \text{in } \mathbf{R}^3 \tag{2.18}$$

admits no solution if $1 < q < 5$, see,[116] it has a unique radially symmetric solution $u = u(|x|)$ if $q \geq 5$. In the case of $q > 5 = \frac{n+2}{n-2}$, it holds that

$$|x|^{\frac{2}{q-1}} u(|x|) \to L = \left[\frac{2}{q-1}\left(n-2-\frac{2}{q-1}\right)\right]^{\frac{1}{q-1}}, \qquad |x| \to \infty, \tag{2.19}$$

see.[126] This relation implies

$$\int_{\mathbf{R}^3} u^q = +\infty$$

and, therefore, (2.3) admits no radially symmetric finite total mass equilibrium in the case of $1 < \gamma < 6/5$.
If $q = 5$, finally, the solution to (2.18) is classified,[37] and it holds that

$$u(x) = \left(\frac{\mu\sqrt{3}}{\mu^2 + |x|^2}\right)^{1/2}$$

with $\mu > 0$ up to the translation. Total mass defined by this solution is thus not finite. Consequently, given $M > 0$, we obtain no radially symmetric equilibrium of (2.3) with the total mass M in the case of $\gamma = 6/5$. In this way, the radially symmetric equilibrium of the Euler-Poisson equation (2.3) has different profiles according to $\gamma \in (1, 6/5)$, $\gamma = 4/3$, $\gamma = 6/5$, $\gamma \in (6/5, 2) \setminus \{4/3\}$, and $\gamma = 4/3$.

The general entire solution to

$$-\Delta v = v^q, \quad v > 0 \qquad \text{in } \mathbf{R}^n \tag{2.20}$$

associated with the critical Sobolev exponent $q = \frac{n+2}{n-2}$ with $n \geq 3$ was studied earlier by,[115] and it is proven that the solution $v = v(x)$ has the form of

$$v_{x_0,\mu}(x) = \frac{[n(n-2)\mu^2]^{(n-2)/4}}{\left(\mu^2 + |x - x_0|^2\right)^{(n-2)/2}} \tag{2.21}$$

for some $x_0 \in \mathbf{R}^n$ and $\mu > 0$, provided that

$$v(x) = O\left(|x|^{2-n}\right) \qquad \text{as } |x| \to \infty. \tag{2.22}$$

In the context of differential geometry, this property means that any metric conformal to the standard metric ds_0 on S^n with the same mean curvature is a pull-back of ds_0 by the conformal transformation on S^n, see.[229]

Similarly to the case $n = 3$ described above, the assumption (2.22) is not necessary to derive (2.21) from (2.20) for $q = \frac{n+2}{n-2}$, see.[37] Furthermore, it is proven also that the solution to (2.20) does not exist in the case of $1 < q < \frac{n+2}{n-2}$, see.[116] The latter result guarantees *a priori* bounds of the solution to the elliptic and parabolic problems with sub-critical nonlinearity defined on the bounded domain, see[117,119] and §§2.3.3 and 2.4.1. This property is actually proven by the blowup analysis, and §2.3.3 describes the details of the elliptic case.

Although radial symmetry of the general solution to (2.20) does not follow in the super-critical case $q > \frac{n+2}{n-2}$, see,[127,356] the total set of entire radial solutions has remarkable structures, and relation (2.19) is just an example of this significance. See[222] and the references therein.

Variational Equilibrium

We re-formulate the equilibrium using the calculus of variation. More precisely, the total energy is reduced to

$$\mathscr{F}(\rho) = \int_{\mathbf{R}^3} \frac{p}{\gamma - 1} - \frac{1}{2}\langle \Gamma * \rho, \rho \rangle$$

in the case of $v = 0$, and this (variational) equilibrium is defined by $v = 0$ and $\delta \mathscr{F}(\rho) = 0$, under the constraint of

$$\rho \geq 0, \qquad \int_{\mathbf{R}^3} \rho = M.$$

We obtain the semi-unfolding

$$E(v, \rho) \geq \mathscr{F}(\rho)$$

by the defintition, and, therefore, the linearized stability of the variational equilibrium implies its dynamical stability naturally, see §2.1.4. The above functional, furthermore, has the form

$$\mathscr{F} = \frac{A\gamma}{\gamma - 1} F^* - G^*$$

for

$$F^*(\rho) = \frac{1}{\gamma}\int_{\mathbf{R}^3} \rho^\gamma, \qquad \rho \geq 0$$
$$G^*(\rho) = \frac{1}{8\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{\rho(x)\rho(x')}{|x-x'|}dxdx', \quad \int_{\mathbf{R}^3}\rho = M,$$

and hence we can develop the theory of Toland duality.
To realize these functionals, F^* and G^*, as the Legendre transformation of proper, convex, lower semi-continuous functionals, first, we take $X = \dot{H}^1(\mathbf{R}^3)\oplus\mathbf{R}$ and define

$$G(\mu) = \frac{1}{2}\|\nabla\xi\|_2^2 + Mc$$

for $\mu = \xi \oplus c \in X$, where

$$\dot{H}^1(\mathbf{R}^3) = \left\{\mu \in L^6(\mathbf{R}^3) \mid \nabla\mu \in L^2(\mathbf{R}^3)\right\}.$$

This functional is proper, convex, lower semi-continuous, and it holds that

$$\rho \in \partial G(\mu) \qquad \Leftrightarrow \qquad -\Delta\mu = \rho \ \text{ in } \mathbf{R}^3, \ \int_{\mathbf{R}^3}\rho = M$$

for $(\mu,\rho)\in X\times X^*$ and furthermore,

$$\begin{aligned} G^*(\rho) &= \sup_{\mu\in X}\{\langle\mu,\rho\rangle - G(\mu)\} \\ &= \sup_{\xi\in\dot{H}^1(\mathbf{R}^3),\ c\in\mathbf{R}} \left\{\langle\xi,\rho\rangle - \frac{1}{2}\|\nabla\xi\|_2^2 + c\langle 1,\rho\rangle - Mc\right\} \\ &= \frac{1}{2}\left\langle(-\Delta)^{-1}\rho,\rho\right\rangle + \chi_{\{\langle\rho,1\rangle=M\}} \end{aligned}$$

for $\rho\in X^*$, under the agreement that $\Delta : \dot{H}^1(\mathbf{R}^3)\to\dot{H}^1(\mathbf{R}^3)^*$ is an isomorphism and $X^*\hookrightarrow \dot{H}^1(\mathbf{R}^3)^*$ by $\dot{H}^1(\mathbf{R}^3)\hookrightarrow X$.
Next, we define

$$F(\mu) = \frac{\gamma-1}{\gamma}\int \xi_+^{\frac{\gamma}{\gamma-1}}$$

for $\mu = \xi\oplus c\in X$, which is also proper, convex, lower semi-continuous. Then we obtain

$$\rho\in\partial F(\mu) \qquad \Leftrightarrow \qquad \xi_+ = \rho^{\gamma-1}$$

for $(\mu,\rho)\in X\times X^*$ with $\mu = \xi\oplus c$, and, in particular, $0\leq\rho\in L^\gamma(\mathbf{R}^3)$ follows in this case. These results are summarized by

$$\begin{aligned} F^*(\rho) &= \sup_{\mu\in X}\{\langle\mu,\rho\rangle - F(\mu)\} \\ &= \begin{cases} \frac{1}{\gamma}\int_{\mathbf{R}^3}\rho^\gamma, & \rho\geq 0 \\ +\infty, & \text{otherwise} \end{cases} \end{aligned}$$

for $\rho\in X^*$.

Variational equilibrium is now formulated by

$$\frac{A\gamma}{\gamma-1}\partial F^*(\rho)\cap\partial G^*(\rho)\neq\emptyset$$

with $\rho\in X^*$, and thus it holds that

$$-\Delta\mu=\rho \ \text{ in } \mathbf{R}^3, \qquad \rho=\left[\frac{\gamma-1}{A\gamma}\xi\right]_+^{1/(\gamma-1)}$$
$$\int_{\mathbf{R}^3}\rho=M$$

for

$$\mu=\xi\oplus c\in\frac{A\gamma}{\gamma-1}\partial F^*(\rho)\cap\partial G^*(\rho).$$

This relation implies

$$-\Delta\xi=\left[\frac{\gamma-1}{A\gamma}\xi\right]_+^q \quad \text{in } \mathbf{R}^3$$
$$\int_{\mathbf{R}^3}\left[\frac{\gamma-1}{A\gamma}\xi\right]_+^q dx=M \tag{2.23}$$

with $q=1/(\gamma-1)$, and, therefore, the variational equilibrium is defined by

$$\rho=\left[\frac{\gamma-1}{A\gamma}\xi\right]_+^q,$$

using the solution $\xi=\xi(x)$ to (2.23).
Putting $\xi=\alpha u$ and $\alpha=\left(\frac{\gamma-1}{A\gamma}\right)^{\frac{1}{2-\gamma}}$, we obtain (2.13) for

$$\Omega=\left\{x\in\mathbf{R}^3 \mid u(x)>0\right\},$$

and thus, a variational equilibrium is a classical equilibrium. This $\xi=\xi(x)$ is defined on the whole space, has the $C^{2,\theta}$-regularity, and can be negative somewhere. Therefore, the construction of the solution $\xi=\xi(x)$ to (2.23) from that of $\rho=\rho(x)$ of (2.12) or $u=u(x)$ of (2.13) will be difficult, and, therefore, (2.23) is regarded as a different description of the equilibrium to (2.3).
Problem (2.23) arises also in the plasma confinement as we are describing, and furthermore, uniform boundedness and radial symmetry of the solution $\xi=\xi(x)$ to (2.23) with bounded Morse indices are known for $1<q<(n+2)/(n-2)$ if $n\geq 4$ and for $2<q<5$ if $n=3$, see.[135,136]

2.1.3. Plasma Confinement

Given a bounded domain $\Omega \subset \mathbf{R}^3$ with smooth boundary $\partial\Omega$, we consider

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho v) = 0 \\ &\rho\left(v_t + (v\cdot\nabla)v\right) + \nabla p + \rho\nabla\Phi = 0 \\ &\Delta\Phi = \rho, \ \ p = A\rho^\gamma \qquad && \text{in } \Omega\times(0,T) \\ &v\cdot\nu = 0, \ \ \Phi = 0 && \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{2.24}$$

and obtain the non-negativity, total mass conservation, and total energy conservation indicated by

$$\begin{aligned} &\rho \geq 0, \qquad \int_\Omega \rho = M \\ &E = \int_\Omega \frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\,dx - \frac{1}{2}\left\langle(-\Delta_D)^{-1}\rho,\rho\right\rangle \end{aligned} \tag{2.25}$$

similarly. In this case, the variational equilibrium is formulated by $v = 0$ and $\delta\mathscr{F}(\rho) = 0$, where

$$\mathscr{F}(\rho) = \int_\Omega \frac{p}{\gamma-1} - \frac{1}{2}\left\langle(-\Delta_D)^{-1}\rho,\rho\right\rangle$$

defined for

$$\rho \geq 0, \qquad \int_\Omega \rho = M.$$

This functional is nothing but the Berestycki-Brezis functional[21] concerning the plasma confinement. Damlamian[71] observed the Toland duality in this free boundary problem between the above described formulation and that of Temam,[320] where the Nehari principle is involved in the latter case.[219,220]

First, the problem

$$\begin{aligned} &-\Delta v = v_+^q && \text{in } \Omega \\ &v = \text{constant} && \text{on } \Gamma = \partial\Omega \\ &\int_\Omega v_+^q = \lambda \end{aligned} \tag{2.26}$$

arises in plasma confinement, see,[319] where $\Omega \subset \mathbf{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary $\partial\Omega$, $q > 1$, and $\lambda > 0$ is a constant. We note that the above $v = v(x)$ is a scalar function and is different from $v = v(x,t)$ in (2.24). Then, the associated variational functions[21,320] are defined by

$$\begin{aligned} &J(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{q+1}\int_\Omega v_+^{q+1} + \lambda v_\Gamma, \quad v\in H_c^1(\Omega), \int_\Omega v_+^q = \lambda \\ &J^*(u) = \frac{q}{q+1}\int_\Omega u^{\frac{q+1}{q}} - \frac{1}{2}\left\langle(-\Delta_D)^{-1}u,u\right\rangle, \ u \geq 0, \|u\|_1 = \lambda, \end{aligned} \tag{2.27}$$

where

$$H_c^1(\Omega) = \left\{v\in H^1(\Omega) \mid v = \text{constant on } \Gamma\right\}.$$

Henceforth, we describe only the sub-critical case of $1 < q < \frac{n+2}{(n-2)_+}$. Then, the solution $v = v(x)$ to (2.26) is $C^{2,\theta}$ on $\overline{\Omega}$ from the elliptic regularity, and the Toland duality is examined similarly to the the whole space case described in §2.1.2. As is confirmed there, the stationary problem (2.38) with $q = 4/3$ is scaling invariant and actually is provided with the mass quantization, see §2.4.3.

Field Variation

First, putting

$$G(v) = \frac{1}{2}\|\nabla v\|_2^2 + \lambda v_\Gamma$$
$$F(v) = \frac{1}{q+1}\int_\Omega v_+^{q+1}$$

for $v \in X = H_c^1(\Omega)$, we obtain

$$\begin{aligned} u \in \partial G(v) &\Leftrightarrow (\nabla v, \nabla w) + \lambda w_\Gamma = \langle w, u\rangle, \qquad w \in X \\ &\Leftrightarrow \langle 1, u\rangle = \lambda,\ \ v - v_\Gamma = (-\Delta_D)^{-1} u \end{aligned} \tag{2.28}$$

and

$$u \in \partial F(v) \quad \Leftrightarrow \quad u = v_+^q \tag{2.29}$$

for $(u, v) \in X \times X^*$. Thus, the problem (2.26) is equivalent to

$$u \in \partial F(v) \cap \partial G(v),$$

or

$$\delta J(v) = 0, \qquad u = v_+^q,$$

because

$$u = v_+^q, \qquad v \in X$$

implies

$$0 \le u \in L^{\frac{q+1}{q}} \hookrightarrow X^*.$$

It holds that $D(J) = X$ and

$$\begin{aligned} J(v) &= G(v) - F(v) \\ &= \frac{1}{2}\|\nabla v\|_2^2 + \lambda v_\Gamma - \frac{1}{q+1}\int_\Omega v_+^{q+1}, \end{aligned}$$

and, therefore, the constraint $\int_\Omega v_+^q = \lambda$ is superfluous to derive (2.26). This property means that the Nehari principle, see,[303] is involved in the original formulation (2.27).

Particle Variation

Given $u \in X^*$, we obtain

$$\begin{aligned} G^*(u) &= \sup_{v\in X}\{\langle v,u\rangle - G(v)\} \\ &= \sup_{v\in X}\left\{\langle v-v_\Gamma,u\rangle + v_\Gamma\langle 1,u\rangle - \frac{1}{2}\|\nabla(v-v_\Gamma)\|_2^2 - \lambda v_\Gamma\right\} \\ &= \chi_{\{\langle 1,u\rangle=\lambda\}} + \sup_{v\in H_0^1(\Omega)}\left\{\langle v,u\rangle - \frac{1}{2}\|\nabla v\|_2^2\right\} \\ &= \chi_{\{\langle 1,u\rangle=\lambda\}} + \frac{1}{2}\left\langle(-\Delta_D)^{-1}u,u\right\rangle \end{aligned}$$

and

$$\begin{aligned} F^*(u) &= \sup_{v\in X}\{\langle v,u\rangle - F(v)\} \\ &= \begin{cases} \frac{q}{q+1}\int_\Omega u^{\frac{q+1}{q}}, & 0\le u\in L^{\frac{q+1}{q}}(\Omega) \\ +\infty, & \text{otherwise} \end{cases} \end{aligned}$$

because

$$u\in\partial F(v) \;\Leftrightarrow\; v\in\partial F^*(u) \;\Leftrightarrow\; u=v_+^q$$

for $(u,v)\in X\times X^*$. Thus the "free energy" is defined by

$$\begin{aligned} D(J^*) &= D(G^*)\cap D(F^*) \\ &= \left\{u\in X^* \mid 0\le u\in L^{\frac{q+1}{q}}(\Omega),\ \ \|u\|_1=\lambda\right\} \end{aligned}$$

and

$$J^*(u) = \begin{cases} \frac{q}{q+1}\int_\Omega u^{\frac{q+1}{q}} - \frac{1}{2}\left\langle(-\Delta_D)^{-1}u,u\right\rangle, & u\in D(J^*) \\ +\infty, & \text{otherwise,} \end{cases}$$

and this structure of variation is nothing but Berestycki-Brezis' formulation to (2.26).

Duality

If the critical point $\overline{v}=\overline{v}(x)$ of the Temam functional $J=J(v)$ defined on $X=H_c^1(\Omega)$ is not positive-definite on $\overline{\Omega}$, then the mapping $v\mapsto u$ defined by $u\in\partial F(v)$ around $v=\overline{v}$, that is (2.29) is not faithful. The relation $u\in\partial G(v)$, on the other hand, induces an isomorphism between $u\in X^*$ with $\langle 1,u\rangle=\lambda$ and $v-v_\Gamma\in H_0^1(\Omega)$ for $v\in X=H_c^1(\Omega)$. Even in this case, the Nehari constraint $\int_\Omega v_+^q=\lambda$ is not efficient to assign v_Γ from $v-v_\Gamma$ if supp $v\subset\Omega$. Thus, the argument in §1.2.5 does not work, but still we have "semi-spectral equivalence" using the above isomorphism, as is described in the following paragraph.

2.1.4. Stable Equilibrium

This paragraph is devoted to the stability of the variational equilibrium defined for (2.24). Although the Euler-Poisson equation (2.3) is not formulated by the gradient system, its equilibrium is associated with the duality between the Plasma confinement problem, and the structure of semi-duality controls its dynamical stability.
To begin with, again, we confirm the abstract theory of Toland duality described in §1.2.6; X denotes a Banach space over $\mathbf{R}$, $F,G : X \to (-\infty,+\infty]$ are proper, convex, lower semi-continuous, and

$$L(u,v) = F^*(u) + G(v) - \langle v,u \rangle$$

is the Lagrangian. It holds that

$$\begin{aligned}
J^*(u) &= \inf_{v\in X} L(u,v) = F^*(u) - G^*(u) = L(u,\overline{v}) \\
J(v) &= \inf_{u\in X^*} L(u,v) = G(v) - F(v) = L(\overline{u},v) \\
D(J^*) &= D(F^*) \cap D(G^*) \subset X^* \\
D(J) &= D(G) \cap D(F) \subset X
\end{aligned}$$

for $\overline{v} \in \partial G^*(u)$ and $\overline{u} \in \partial F(v)$, that $\delta J(\overline{v}) = 0$ and $\delta J^*(\overline{u}) = 0$ mean $\partial G(\overline{v}) \cap \partial F(\overline{v}) \neq \emptyset$ and $\partial G(\overline{u}) \cap \partial F^*(\overline{u}) \neq \emptyset$, respectively, and that

$$\overline{u} \in \partial G(\overline{v}) \cap \partial F(\overline{v}) \quad \Leftrightarrow \quad \overline{v} \in \partial G^*(\overline{u}) \cap \partial F^*(\overline{v}). \tag{2.30}$$

If one of the conditions of (2.30) is satisfied, then

$$J^*(\overline{u}) = L(\overline{u},\overline{v}) = J(\overline{v})$$

follows, and, furthermore, if $u \in \partial G(v)$, or equivalently $v \in \partial G^*(u)$, it holds that

$$J^*(u) = L(u,v) \leq J(v).$$

Thus, we obtain

$$u \in \partial G(v) \quad \Rightarrow \quad J^*(u) - J^*(\overline{u}) \leq J(v) - J(\overline{v}), \tag{2.31}$$

and similarly,

$$u \in \partial F(v) \quad \Rightarrow \quad J(v) - J(\overline{v}) \leq J^*(u) - J^*(\overline{u}). \tag{2.32}$$

Temam Functional

Turning to the plasma confinement (2.26), we note that Temam's functional $J(v)$ of (2.27) is twice-differentiable by $q > 1$. If the constraint $\int_\Omega v_+^q = \lambda$ is taken into account, then the linearized stability of a critical point $\overline{v} \in X = H_c^1(\Omega)$ of $J = J(v)$ is defined by the positivity of the self-adjoint operator in $L^2(\Omega)$ associated with the quadratic form on $X_0 \times X_0$ defined by

$$\begin{aligned} Q(\varphi,\varphi) &= \frac{1}{2}\frac{d^2}{ds^2}J(\overline{v}+s\varphi)\bigg|_{s=0} \\ &= \|\nabla\varphi\|_2^2 - \int_\Omega c(x)\varphi^2, \end{aligned}$$

where $c(x) = qv_+^{q-1}$ and

$$X_0 = \left\{\varphi \in X \mid \int_\Omega c(x)\varphi = 0\right\}.$$

This condition means

$$\inf\left\{Q(\varphi,\varphi) \mid \varphi \in H_c^1(\Omega),\ \int_\Omega c(x)\varphi = 0,\ \|\varphi\|_2 = 1\right\} > 0 \tag{2.33}$$

and, then, there is $\varepsilon_0 > 0$ such that each $\varepsilon \in (0,\varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{l} \|v-\overline{v}\|_X < \varepsilon_0 \\ \int_\Omega v_+^q = \lambda \\ J(v) - J(\overline{v}) < \delta \end{array} \quad \Rightarrow \quad \|v-\overline{v}\|_X < \varepsilon.$$

Using (2.32), we see that if $\overline{v} \in X$ is a linearly stable critical point of J and $\overline{u} = \overline{v}_+^q$, then there is $\varepsilon_0 > 0$ such that each $\varepsilon \in (0,\varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{l} u = v_+^q \\ v \in X = H_c^1(\Omega) \\ \int_\Omega v_+^q = \lambda \\ \|v-\overline{v}\|_X < \varepsilon_0 \\ J^*(u) - J^*(\overline{u}) < \delta \end{array} \quad \Rightarrow \quad \|v-\overline{v}\|_X < \varepsilon. \tag{2.34}$$

A different linearized stability is obtained by the positivity of the self-adjoint operator in $L^2(\Omega)$ associated with the quadratic form on $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$Q(\varphi,\varphi) = \|\nabla\varphi\|_2^2 - \int_\Omega c(x)\varphi^2.$$

This condition means

$$\inf\left\{Q(\varphi,\varphi) \mid \varphi \in H_0^1(\Omega),\ \|\varphi\|_2 = 1\right\} > 0 \tag{2.35}$$

and then it follows that

$$\begin{array}{l} u = v_+^q \\ v \in X = H_c^1(\Omega) \\ v_\Gamma = \overline{v}_\Gamma \\ \|v-\overline{v}\|_X < \varepsilon_0 \\ J^*(u) - J^*(\overline{u}) < \delta \end{array} \quad \Rightarrow \quad \|v-\overline{v}\|_X < \varepsilon. \tag{2.36}$$

Berestycki-Brezis Functional

The variational equilibrium, on the other hand, is defined by the functional

$$J^*(\rho) = \int_\Omega \frac{p}{\gamma - 1} - \frac{1}{2}\left\langle (-\Delta_D)^{-1}\rho, \rho \right\rangle,$$

in the Euler-Poisson equation (2.24) with $n = 3$. For later convenience, we formulate this functional in a slightly different manner from the whole space case, that is $J^* = F^* - G^*$, where

$$D(F^*) = \left\{ \rho \in L^{\frac{q+1}{q}}(\Omega) \mid \rho \geq 0 \right\}$$
$$F^*(\rho) = \begin{cases} \frac{A}{\gamma-1} \cdot \frac{q+1}{q} \cdot \frac{q}{q+1} \int_\Omega \rho^{\frac{q+1}{q}}, & \rho \in D(F^*) \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$D(G^*) = \{\rho \in X^* \mid \langle 1, \rho \rangle = M\}$$
$$G^*(\rho) = \frac{1}{2}\left\langle (-\Delta_D)^{-1}\rho, \rho \right\rangle$$

for

$$\rho \in X^* \subset H^{-1}(\Omega) = H_0^1(\Omega)^*$$
$$q = \frac{1}{\gamma - 1} \in (1, 5).$$

Since

$$\frac{A}{\gamma - 1} \cdot \frac{q+1}{q} \rho^{\frac{q+1}{q}} = (B\rho)^{\frac{q+1}{q}}$$
$$B = \left(\frac{A\gamma}{\gamma - 1} \right)^{1/\gamma},$$

it holds that

$$\begin{aligned} F(\mu) &= \sup_{\rho \in X^*} \left\{ \left\langle B^{-1}\rho, B\rho \right\rangle - F^*(\rho) \right\} \\ &= \frac{1}{q+1} \int_\Omega \left(B^{-1}\mu_+ \right)^{q+1} \end{aligned}$$

and, therefore, the dual functional is defined by

$$\begin{aligned} J(\mu) &= G(\mu) - F(\mu) \\ &= \frac{1}{2}\|\nabla \mu\|_2^2 + M\mu_\Gamma - \frac{a}{q+1}\int_\Omega \mu_+^{q+1} \end{aligned} \tag{2.37}$$

for $\mu \in X = H^1_c(\Omega)$, where $a = (B^{-1})^{q+1} = \left(\frac{\gamma-1}{A\gamma}\right)^{1/(\gamma-1)}$. Using this functional, the variational equilibrium of (2.24) is now defined by $v = 0$ and $\rho = a\mu^q_+$, where

$$\begin{aligned} -\Delta\mu &= a\mu^q_+ \qquad \text{in } \Omega \\ \mu &= \text{constant} \qquad \text{on } \partial\Omega \\ \int_\Omega a\mu^q_+ &= M. \end{aligned} \tag{2.38}$$

Let $\mu = \overline{\mu}(x)$ be a solution to (2.38), and $(v,\rho) = (0,\overline{\rho})$ be the equilibrium of (2.24) defined by $\overline{\rho} = a\overline{\mu}^q_+$. Then, the total energy (2.24) is described by

$$E(v,\rho) = \int_\Omega \frac{\rho}{2}|v|^2\,dx + J^*(\rho),$$

and hence it holds that

$$\begin{aligned} E(v_0,\rho_0) - E(0,\overline{\rho}) &= E(v,\rho) - E(0,\overline{\rho}) \\ &\geq J^*(\rho) - J^*(\overline{\rho}) \end{aligned} \tag{2.39}$$

for $(v_0,\rho_0) = (v,\rho)|_{t=0}$. Therefore, applying (2.34) and (2.36), we obtain the following theorems concerning the cases that the plasma region occupies $\overline{\Omega}$ and is enclosed in $\overline{\Omega}$, respectively.

Theorem 2.1. *Let $6/5 < \gamma < 2$ and $\overline{\mu} = \overline{\mu}(x)$ be a solution to (2.38), positive on $\overline{\Omega}$, and linearly stable in the sense of (2.33) for $c(x) = aq\overline{\mu}^{q-1}$. Then, each $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\begin{aligned} &\|v_0\|_2 + \left\|\rho_0^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1} < \delta \\ &\rho_0 \geq 0, \qquad \int_\Omega \rho_0 = M \\ &\Rightarrow \\ &\sup_{t\in[0,T)} \left\{\|v(\cdot,t)\|_2 + \left\|\rho(\cdot,t)^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1}\right\} < \varepsilon, \end{aligned}$$

where $(v,\rho) = (v(\cdot,t),\rho(\cdot,t))$ is a solution to (2.24) satisfying (2.25) and

$$\rho(\cdot,t)^{\gamma-1} \in C\left([0,T), H^1(\Omega)\right), \tag{2.40}$$

$\overline{\rho} = a\overline{\mu}^q$, and $T > 0$ is arbitrary.

Theorem 2.2. *Let $5/6 < \gamma < 2$ and $\overline{\mu} = \overline{\mu}(x)$ be a solution to (2.38), supp $\overline{\mu} \subset \Omega$, and linearly stable in the sense of (2.35) for $c(x) = aq\overline{\mu}^{q-1}_+$. Then, each $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\begin{aligned} &\|v_0\|_2 + \left\|\rho_0^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1} < \delta \\ &\rho_0 \geq 0, \qquad \mathit{supp}\ \rho_0^{\gamma-1} \subset \Omega \\ &\Rightarrow \\ &\sup_{t\in[0,T)} \left\{\|v(\cdot,t)\|_2 + \left\|\rho(\cdot,t)^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1}\right\} < \varepsilon \\ &\mathit{supp}\ \rho(\cdot,t)^{\gamma-1} \subset \Omega, \end{aligned}$$

where $(v,\rho) = (v(\cdot,t),\rho(\cdot,t))$ *is a solution to (2.24) satisfying (2.25) and (2.40),* $\overline{\rho} = a\overline{\mu}_+^q$*, and* $T > 0$ *is arbitrary.*

2.1.5. Related Problems

This paragraph is devoted to several remarks on the Euler-Poisson equation and the plasma confinement. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$.

Neumann Case

In the case that $X = H^1(\Omega)$, $1 < q < 5$, and

$$\begin{aligned} G(v) &= \frac{1}{2}\|\nabla v\|_2^2 + \frac{\lambda}{|\Omega|}\int_\Omega v \\ F(v) &= \frac{1}{q+1}\int_\Omega v_+^{q+1}, \end{aligned}$$

it holds that

$$\begin{aligned} u \in \partial G(v) &\Leftrightarrow (\nabla v, \nabla w) + \frac{\lambda}{|\Omega|}\int_\Omega w = \langle w, u\rangle, \qquad w \in X \\ &\Leftrightarrow \langle 1, u\rangle = \lambda,\ v - \frac{1}{|\Omega|}\int_\Omega v = (-\Delta_{JL})^{-1}u. \end{aligned}$$

Introducing the anti-free energy by

$$J(v) = G(v) - F(v)$$

with $D(J) = X$, we obtain $\delta J(v) = 0$, or equivalently $u \in \partial G(v) \cap \partial F(v)$ with $u \in X^*$, if and only if

$$\begin{aligned} &u = v_+^q \\ &v \in X = H^1(\Omega) \\ &v - \frac{1}{|\Omega|}\int_\Omega v = (-\Delta_{JL})^{-1}u. \end{aligned}$$

Thus $\delta J(v) = 0$ means

$$\begin{aligned} &-\Delta v = v_+^q - \frac{\lambda}{|\Omega|} \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \\ &\int_\Omega v = 0. \end{aligned} \tag{2.41}$$

Given $u \in X^*$, we obtain

$$\begin{aligned}
G^*(u) &= \sup_{v\in X}\{\langle v,u\rangle - G(v)\}\\
&= \sup_{v\in X}\{\langle v - \frac{1}{|\Omega|}\int_\Omega v, u\rangle + \frac{1}{|\Omega|}\int_\Omega v\cdot\langle 1,u\rangle - \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda}{|\Omega|}\int_\Omega v\}\\
&= \sup_{v\in X,\ \int_\Omega v=0}\{\langle v,u\rangle - \frac{1}{2}\|\nabla v\|_2^2\} + 1_{\{\langle 1,u\rangle=\lambda\}}(u)\\
&= 1_{\{\langle 1,u\rangle=\lambda\}}(u) + \frac{1}{2}\langle(-\Delta_{JL})^{-1}u,u\rangle,
\end{aligned}$$

and, therefore, the dual variation to (2.41) is defined by

$$\begin{aligned}
D(J^*) &= \left\{u\in X^* \mid 0\le u\in L^{\frac{q+1}{q}}(\Omega),\ \ \|u\|_1=\lambda\right\}\\
J^*(u) &= \begin{cases} \frac{q}{q+1}\int_\Omega u^{\frac{q+1}{q}} - \frac{1}{2}\left\langle(-\Delta_{JL})^{-1}u,u\right\rangle, & u\in D(J^*)\\ +\infty, & \text{otherwise}\end{cases}
\end{aligned}$$

The associated Euler-Poisson equation is described by

$$\begin{aligned}
&\rho_t + \nabla\cdot(v\rho) = 0\\
&\rho\,(v_t + (v\cdot\nabla)v) + \nabla p + \rho\nabla\Phi = 0\\
&\Delta\Phi = \rho - \frac{1}{|\Omega|}\int_\Omega \rho,\ \ p = A\rho^\gamma && \text{in } \Omega\times(0,T)\\
&\nu\cdot v = 0,\ \ \frac{\partial\Phi}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T)\\
&\int_\Omega \Phi = 0, && 0<t<T
\end{aligned} \tag{2.42}$$

for $6/5<\gamma<2$. We obtain

$$\begin{aligned}
&\rho\ge 0, \qquad \int_\Omega \rho = M\\
&E = \int_\Omega \frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\,dx - \frac{1}{2}\left\langle(-\Delta_{JL})^{-1}\rho,\rho\right\rangle,
\end{aligned} \tag{2.43}$$

and the variational equilibrium is derived from

$$J^*(\rho) = \int_\Omega \frac{p}{\gamma-1} - \frac{1}{2}\left\langle(-\Delta_{JL})^{-1}\rho,\rho\right\rangle,$$

that is, $\rho = a\mu_+^q$ for

$$\begin{aligned}
&-\Delta\mu = a\mu_+^q - \frac{M}{|\Omega|} \quad \text{in } \Omega, \qquad \frac{\partial\mu}{\partial\nu} = 0 \quad \text{on } \partial\Omega\\
&\int_\Omega \mu = 0,
\end{aligned} \tag{2.44}$$

where $a = \left(\frac{\gamma-1}{A\gamma}\right)^{1/(\gamma-1)}$.

The linearized stability of this equilibrium is defined by

$$\inf\left\{ Q(\varphi,\varphi) \mid \varphi \in H^1(\Omega),\ \int_\Omega \varphi = 0,\ \|\varphi\|_2 = 1 \right\} > 0, \tag{2.45}$$

where

$$\begin{aligned} Q(\varphi,\varphi) &= \|\nabla\varphi\|_2^2 - \int_\Omega c(x)\varphi^2 \\ c(x) &= aq\mu_+^{q-1}, \end{aligned}$$

and in this case we obtain the following theorem.

Theorem 2.3. *Let $6/5 < \gamma < 2$ and $\overline{\mu} = \overline{\mu}(x)$ be a solution to (2.44), linearly stable in the sense of (2.45) for $c(x) = aq\overline{\mu}_+^{q-1}$. Then, each $\varepsilon > 0$ admits $\delta > 0$ such that*

$$\begin{aligned} &\|v_0\|_2 + \left\|\rho_0^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1} < \delta \\ &\rho_0 \geq 0,\ \int_\Omega \rho_0 = M \\ &\Rightarrow \\ &\sup_{t\in[0,T)} \left\{ \|v(\cdot,t)\|_2 + \left\|\rho(\cdot,t)^{\gamma-1} - \overline{\rho}^{\gamma-1}\right\|_{H^1} \right\} < \varepsilon, \end{aligned}$$

where $(v,\rho) = (v(\cdot,t),\rho(\cdot,t))$ is a solution to (2.42) satisfying (2.43) and (2.40), $\overline{\rho} = a\overline{\mu}_+^q$, and $T > 0$ is arbitrary.

Navier-Stokes-Poisson Equation

There are several technical difficulties in mathematical study of the Euler equation for both compressible and incompressible cases.[186,187] Theorems 2.1-2.3 also are not satisfactory because well-posedness of the problem is only established in $H^s(\mathbf{R}^3)^4$ with $s > 5/2$ for $(\rho^{(\gamma-1)/2}, v)$. Obviously, this function space has a serious discrepancy in the function space describing stability, $H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)^3$ for $(\rho^{\gamma-1}, v)$. We can construct, on the other hand, weak solutions to the Navier-Stokes-Poisson equation in more moderate function spaces by the method of compensated compactness.[94]
This system is associated with the Lamé constants $\lambda > 0$ and σ in $\sigma + \frac{2}{3}\lambda \geq 0$ and is described by

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho v) = 0 \\ &\rho\,(v_t + (v\cdot\nabla)v) + \rho\nabla\Phi + \nabla p = \lambda\Delta v + (\sigma+\lambda)\nabla(\nabla\cdot v) \\ &\Delta\Phi = \rho - \frac{1}{|\Omega|}\int_\Omega \rho,\ \ p = A\rho^\gamma && \text{in } \Omega\times(0,T) \\ &v = 0,\ \ \frac{\partial\Phi}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T) \\ &\int_\Omega \Phi = 0, && 0 < t < T. \end{aligned} \tag{2.46}$$

In this case, if $\frac{3}{2} < \gamma < 2$, then there is a solution,[81,172] denoted by $(v,\rho) = (v(\cdot,t),\rho(\cdot,t))$, satisfying

$$0 \le \rho \in C_*\left([0,+\infty), L^\gamma(\Omega)\right)$$
$$v \in L^2_{loc}\left(0,+\infty; H^1_0(\Omega)\right)$$
$$\rho v \in C_*\left([0,+\infty), L^{\frac{2\gamma}{\gamma+1}}(\Omega)\right)$$

such that

$$E = \int_\Omega \left(\frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\right) - \frac{1}{2}\left\langle(-\Delta_{JL})^{-1}\rho,\rho\right\rangle \in W^{1,1}_{loc}[0,T)$$
$$\frac{dE}{dt} + \lambda\|\nabla v\|_2^2 + (\sigma+\lambda)\|\nabla\cdot v\|_2^2 \le 0$$
$$\int_\Omega \rho(\cdot,t) = M,$$

and the additional properties which comprise of the *renormalized solution*.[172]
We formulate the variational equilibrium, using

$$J^*(\rho) = \int_\Omega \frac{p}{\gamma-1} - \frac{1}{2}\left\langle(-\Delta_{JL})^{-1}\rho,\rho\right\rangle$$

defined for

$$0 \le \rho \in L^\gamma(\Omega), \qquad \int_\Omega \rho = M. \tag{2.47}$$

Although this functional is not twice-differentiable, we define the linearlized stability of its critical function $\overline{\rho} = \overline{\rho}(x)$ by the existence of $\varepsilon_0 > 0$ such that each $\varepsilon \in (0,\varepsilon_0/4]$ admits $\delta > 0$ such that

$$\begin{array}{l} \|\rho-\overline{\rho}\|_\gamma < \varepsilon_0 \\ \int_\Omega \rho = \int_\Omega \overline{\rho} \\ J^*(\rho) - J^*(\overline{\rho}) < \delta \end{array} \quad\Rightarrow\quad \|\rho-\overline{\rho}\|_\gamma < \varepsilon \tag{2.48}$$

for $\rho = \rho(x)$ satisfying (2.47).
If

$$\limsup_{s\downarrow t} \|\rho(\cdot,s)\|_\gamma \le \|\rho(\cdot,t)\|_\gamma \tag{2.49}$$

for any $t \ge 0$, then $t \in [0,+\infty) \mapsto \rho(\cdot,t) \in L^\gamma(\Omega)$ is right-continuous. In this case, such $\overline{\rho}$ is dynamically stable and each $\varepsilon >$ admits $\delta > 0$ such that

$$\begin{array}{l} \rho_0 \ge 0 \\ \int_\Omega \rho_0 = \int_\Omega \overline{\rho} \\ \|v_0\|_2 + \|\rho_0 - \overline{\rho}\|_\gamma < \delta \end{array} \quad\Rightarrow\quad \sup_{t\ge 0}\left\{\|v(\cdot,t)\|_2 + \|\rho(\cdot,t)-\overline{\rho}\|_\gamma\right\} < \varepsilon.$$

Inequality (2.49) has not been established in the framework of the renormalized solution to (2.46), and furthermore, (2.48) is not able to reduce to a similar problem concerning the twice-differentiable "anti-free energy functional" although the above argument provides a guidline to the classification of the stable stationary state.

Self-Similar Blowup

The Euler-Poisson equation (2.3) with $\gamma = 4/3$ is involved by the quantized blowup mechanism defined by §1.1, and here we describe the situation briefly. First, the "free energy functional" determing the equilibrium to (2.3),

$$\mathscr{F}(\rho) = \int_{\mathbf{R}^3} \frac{p}{\gamma-1} - \frac{1}{2}\langle \Gamma * \rho, \rho \rangle, \qquad \rho \geq 0, \ \int_{\mathbf{R}^3} \rho = M,$$

has the scaling homogeneity in this case of $\gamma = 4/3$. More precisely, it holds that

$$\mathscr{F}(\rho_\mu) = \mu \mathscr{F}(\rho), \qquad \rho_\mu \geq 0, \qquad \int_{\mathbf{R}^3} \rho_\mu = \int_{\mathbf{R}^3} \rho$$

for $\rho_\mu(x) = \mu^3 \rho(\mu x)$ with $\mu > 0$, and this property leads to the quantized blowup mechanism at the equilibrium level.[331,332,340] See[308] and §2.4.3 for the connection to the two-dimensional mass quantization. There is another non-equilibrium associated with this free energy. It is formulated by the degenerate parabolic equation derived from the kinetic theory, see §2.2.6.

Next, radially symmetric solution to (2.3) is described by

$$\rho = \rho(r,t), \quad v = \omega w(r,t), \quad r = |x|, \quad \omega = \frac{x}{r},$$

and then it holds that

$$\begin{aligned} &\rho_t + w\rho_r + \rho w_r + \frac{2}{r}\rho w = 0 \\ &\rho(w_t + \omega w) + p_r + \frac{\rho}{r^2}\int_0^r \rho(s,t) s^2 ds = 0 \\ &p = A\rho^\gamma, \ \gamma = 4/3 \qquad\qquad \text{in } \mathbf{R}^3 \times (0,T). \end{aligned}$$

Using this formulation, we can detect the self-similar solution in the form of

$$\begin{aligned} \rho(r,t) &= \alpha y(r/a(t))^3/a(t)^3 \\ v &= \frac{\dot{a}(t)}{a(t)} x, \end{aligned}$$

where

$$\begin{aligned} &\frac{d^2 a}{dt^2} = -\frac{\beta}{a^2}, \qquad a = a(t) > 0 \\ &\frac{d^2 y}{dx^2} + \frac{2}{r}\frac{dy}{dx} + y^3 = \mu \\ &y|_{x=0} = 1, \quad \left.\frac{dy}{dx}\right|_{x=0} = 0 \end{aligned} \tag{2.50}$$

for $\alpha = \frac{A^{3/2}}{8}$, $\beta > 0$, and $\mu = \frac{3\beta}{8A^{3/2}}$, see.[292] The first equation of (2.50) is provided with the invariant

$$E = \frac{\dot{a}^2}{2} - \frac{\beta}{a},$$

and if $a(0) = a_0 > 0$, $\dot{a}(0) = a_1$, and $a_1 < \sqrt{2\lambda/a_0}$, then this $a = a(t)$ has the extinction time $T > 0$, that is $a(T-0) = 0$. We can show, on the other hand, that the solution $y = y(x)$ to (2.50) has a compact support if $|\mu| \ll 1$ ([194]), and, therefore, $\rho = \rho(r,t)$ develops the delta function singularity in finite time in this case. We note that this self-similar solution is not a stationary solution to the backward self-similar transformaion, and, therefore, its blowup rate is of type (II).

Quantized Blowup Mechanism

The ε-regularity of (2.3) with $\gamma = 4/3$ also holds,[78] and if the total mass is sufficiently small then it holds that

$$\|(\sqrt{\rho}v)(\cdot,t)\|_2 + \|\rho(\cdot,t)\|_\gamma \leq C, \qquad 0 \leq t < T. \tag{2.51}$$

In fact, Sobolev's inequality guarantees

$$\|\nabla v\|_2^2 \geq S_0 \|v\|_6^2$$

for $v \in \dot{H}^1(\mathbf{R}^3) \hookrightarrow L^6(\mathbf{R}^3)$ with $S_0 > 0$, which implies

$$\frac{1}{2}|\langle \Gamma * \rho, \rho \rangle| \leq \frac{1}{2S_0}\|\rho\|_{6/5}^2 \leq \frac{1}{2S_0}\|\rho\|_1^{2/3}\|\rho\|_{4/3}^{4/3}$$

by the duality $L^{6/5}(\mathbf{R}^3) \hookrightarrow \dot{H}^1(\mathbf{R}^3)^*$ and the interpolation

$$\|\rho\|_{6/5} \leq \|\rho\|_1^{1-\theta}\|\rho\|_\gamma^\theta, \qquad \frac{1-\theta}{1} + \frac{\theta}{\gamma} = \frac{5}{6},$$

that is $\theta = \frac{2}{3}$. Using total energy, now we obtain

$$\int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 + \frac{A\rho^\gamma}{\gamma-1}\, dx \leq E + \frac{M^{1/3}}{2S_0}\|\rho\|_\gamma^\gamma,$$

and hence (2.51) in the case of

$$M < \left(\frac{2AS_0}{\gamma-1}\right)^3 = M_*.$$

If (2.51) holds, then it follows that

$$\left|\frac{d^2}{dt^2}\int_{\mathbf{R}^3} \rho\varphi\right| \leq C_\varphi, \qquad 0 \leq t < T$$

from (2.10) for $\varphi \in C_0^\infty(\mathbf{R}^3)$ because

$$\left|(\psi(x) - \psi(x') \cdot \nabla\Gamma(x-x')\right| \leq C_\psi \Gamma(x-x')$$

holds for $\psi \in C_0^\infty(\mathbf{R}^3)^3$. This inequality guarantees the existence of a Radon measure denoted by $\mu(dx,T)$ such that $\mu(\mathbf{R}^3,T) \le M$ and

$$\rho(x,t)dx \rightharpoonup \mu(dx,T).$$

in $\mathscr{M}(\mathbf{R}^3)$ as $t \uparrow T < +\infty$.
Localizing the above profile, first, we modify the proof of (2.4) and deduce

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\varphi\, dx \right.\\
&\qquad \left. -\frac{1}{8\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{\varphi(x)+\varphi(x')}{|x-x'|}\rho(x,t)\rho(x',t)dxdx'\right\}\\
&= \int_{\mathbf{R}^3}\left(\rho|v|^2 + \frac{A\gamma}{\gamma-1}p + \rho\Phi\right)(v\cdot\nabla)\varphi
\end{aligned} \tag{2.52}$$

for $\varphi \in C_0^\infty(\mathbf{R}^3)$. In fact, it holds that

$$\begin{aligned}
&\int_{\mathbf{R}^3}[(\rho v)_t + \nabla\cdot(\rho v\otimes v)]\cdot v\varphi = \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3}|v|^2\rho\varphi - \int_{\mathbf{R}^3}\rho|v|^2(v\cdot\nabla)\varphi\\
&\int_{\mathbf{R}^3}\nabla p\cdot v\varphi = \frac{d}{dt}\int_{\mathbf{R}^3}\frac{p}{\gamma-1}\varphi - \frac{A\gamma}{\gamma-1}\int_{\mathbf{R}^3}\rho^\gamma v\cdot\nabla\varphi\\
&\int_{\mathbf{R}^3}\rho\nabla\Phi\cdot v\varphi = -\int_{\mathbf{R}^3}\Phi\varphi\rho_t - \int_{\mathbf{R}^3}\Phi\rho v\cdot\nabla\varphi\\
&\qquad = -\frac{1}{8\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\varphi(x)+\varphi(x')}{|x-x'|}\rho(x,t)\rho(x',t)dxdx' - \int_{\mathbf{R}^3}\Phi\rho(v\cdot\nabla)\varphi
\end{aligned}$$

similarly.
Then, we obtain

$$\begin{aligned}
&\frac{1}{8\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\varphi(x)+\varphi(x')}{|x-x'|}\rho(x,t)\rho(x',t)dxdx' = \langle\Gamma*\varphi\rho,\rho\rangle\\
&\qquad = \langle\Gamma\varphi\rho,\varphi\rho\rangle + \frac{1}{4\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\varphi(x)(1-\varphi(x'))}{|x-x'|}\rho(x,t)\rho(x',t)dxdx'
\end{aligned}$$

and

$$0 \le \frac{\varphi(x)(1-\varphi(x'))}{|x-x'|}\begin{cases} = 0, & |x'| \le R/2 \ \text{ or } \ |x| \ge R\\ \le 4/R, & |x'| > R/2, \ |x| < R/4\\ \le \frac{1}{|x-x'|}, & R/2 \le |x'| < 2R, \ R/4 \le |x| < R\\ \le 1/R, & |x'| \ge 2R, \ R/4 \le |x| < R,\end{cases}$$

which results in

$$\begin{aligned}
&0 \le \langle\Gamma*\varphi\rho,\rho\rangle\\
&\le \frac{1}{S_0}\|\varphi\rho\|_{6/5}^2 + \frac{M^2}{\pi R} + \frac{1}{4\pi}\iint_{A_R(x_0)\times A_R(x_0)}\frac{\rho(x,t)\rho(x',t)}{|x-x'|}dxdx'\\
&\le \frac{1}{S_0}\|\varphi\rho\|_1^{2/3}\|\varphi\rho\|_{4/3}^{4/3} + \frac{M^2}{\pi R} + \frac{1}{4\pi}\iint_{A_R(x_0)\times A_R(x_0)}\frac{\rho(x,t)\rho(x',t)}{|x-x'|}dxdx'
\end{aligned}$$

for $A_R(x_0) = B(x_0, 2R) \setminus B(x_0, R/4)$.
Now we take

$$0 \le \varphi = \varphi_{x_0,R}(x) \le 1,$$

supported on $\overline{B(x_0,R)}$, equal to 1 on $\overline{B(x_0,R/2)}$, and satisfying

$$|\nabla \varphi| \le C\varphi^{1/2}$$

as in §1.1. If

$$\begin{aligned} &\rho \in C_* \left([0,T), L^{4/3}_{loc}(\mathbf{R}^3)\right) \\ &\limsup_{t \uparrow T} \|v(\cdot,t)\|_{L^\infty(A_R(x_0))} < +\infty, \end{aligned} \tag{2.53}$$

then it holds that

$$\limsup_{t \uparrow T} \|\rho(\cdot,t)\|_{L^{4/3}(\tilde{A}_R(x_0))} < +\infty \tag{2.54}$$

by the first equation of (2.46), where $\tilde{A}_R(x_0) = B(x_0,R) \setminus B(x_0,R/2)$. Under the assumptions of (2.53) and

$$\limsup_{t \uparrow T} \iint_{A_R(x_0) \times A_R(x_0)} \frac{\rho(x,t)\rho(x',t)}{|x-x'|} dx dx' < +\infty, \tag{2.55}$$

thus we obtain

$$\langle \Gamma * \varphi\rho, \rho \rangle \le C \left\{ 1 + \|\varphi^2 \rho\|_1^{2/3} \|\varphi^2 \rho\|_{4/3}^{4/3} \right\}. \tag{2.56}$$

It holds also that

$$\begin{aligned} &\int_{\mathbf{R}^3} \left(\frac{\rho}{2} |v|^2 + \frac{p}{\gamma - 1} \right) \varphi - \frac{1}{8\pi} \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\varphi(x) + \varphi(x')}{|x-x'|} \rho(x,t)\rho(x',t) dx dx' \\ &\ge \frac{1}{2} \left\| \varphi^{1/2} \sqrt{\rho} v \right\|_2^2 + \frac{A}{\gamma - 1} \left\| \varphi^{3/4} \rho \right\|_{4/3}^{4/3} - \frac{1}{S_0} \|\varphi\rho\|_1^{2/3} \left\| \varphi^{3/4} \rho \right\|_{4/3}^{4/3} - C \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\mathbf{R}^3} \left(\rho |v|^2 + \frac{A\gamma}{\gamma - 1} p + \rho \Phi \right) (v \cdot \nabla) \varphi \right| &\le C \left\{ 1 + \left\langle \Gamma * \varphi^{1/2} \rho, \rho \right\rangle \right\} \\ &\le \left\{ 1 + \|\varphi\rho\|_1^{2/3} \left\| \varphi^{3/4} \rho \right\|_{4/3}^{4/3} \right\} \end{aligned}$$

by (2.56). From these relations, it follows that

$$\left\| \varphi^{1/2} \sqrt{\rho} v \right\|_2^2 + \left(\varepsilon_0^{2/3} - \|\varphi\rho\|_1^{2/3} \right) \left\| \varphi^{3/4} \rho \right\|_{4/3}^{4/3} \le C \tag{2.57}$$

with $C > 0$ and $0 < \varepsilon_0 \ll 1$ independent of $t \in [0,T)$.
Similarly, we obtain

$$\left| \frac{d^2}{dt^2} \int_{\mathbf{R}^3} \rho\varphi \right| \le C \left(1 + \|\varphi\rho\|_1 + \left\| \varphi^{3/4} \rho \right\|_{4/3}^{4/3} \right)$$

by (2.10) under the same assumptions, and, therefore,

$$\left(\varepsilon_0^{2/3} - \|\varphi\rho\|_1^{2/3}\right)_+ \left|\frac{d^2}{dt^2}\|\varphi\rho\|_1\right| \le C(1+\|\varphi\rho\|_1).$$

Thus, there is $\delta > 0$ such that

$$\begin{matrix} \|(\varphi\rho)(\cdot,t_0)\|_1 < \varepsilon_0/4 \\ t_0 \in [T-\delta,T) \end{matrix} \quad \Rightarrow \quad \sup_{t\in[t_0,T)} \|(\varphi\rho)(\cdot,t)\|_1 < \varepsilon_0/2,$$

which results in

$$\liminf_{t\uparrow T} \|(\varphi\rho)(\cdot,t)\|_1 < \varepsilon_0/4 \quad \Rightarrow \quad \limsup_{t\uparrow T} \|(\varphi\rho)(\cdot,t)\|_1 < \varepsilon_0/2$$

and

$$\limsup_{t\uparrow T} \left\{ \left\|\varphi^{1/2}(\sqrt{\rho}v)(\cdot,t)\right\|_2^2 + \left\|\varphi^{3/4}\rho(\cdot,t)\right\|_{4/3}^{4/3} \right\} < +\infty.$$

In other words,

$$\mathscr{S} = \bigcap_{R>0} \left\{ x_0 \in \mathbf{R}^3 \mid \limsup_{t\uparrow T} E_R(x_0,t) = +\infty,\ \limsup_{t\uparrow T} H_R(x_0,t) < +\infty \right\}$$

is finite, and

$$\liminf_{t\uparrow T} \|\rho(\cdot,t)\|_{L^1(B(x_0,r))} \ge \varepsilon_0/4$$

holds for any $x_0 \in \mathscr{S}$ and $0 < r \ll 1$, where

$$E_R(x_0,t) = \|(\sqrt{\rho}v)(\cdot,t)\|_{L^2(B(x_0,R))} + \|p(\cdot,t)\|_{L^1(B(x_0,R))}$$
$$H_R(x_0,t) = \|v(\cdot,t)\|_{L^\infty(A_R(x_0))} + \iint_{A_R(x_0)\times A_R(x_0)} \frac{\rho(x,t)\rho(x',t)}{|x-x'|} dxdx'.$$

In spite of several restrictions to $\mathscr{S}$, this result may be a first step to clarify the non-stationary quantized blowup mechanism in higher-dimensions.

Murakami-Nishihara-Hanawa System

System (2.3) with the heat radiation taken into account arises as a hydrodynamical approximation in the theory of star formation. It is described by

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho v) = 0 \\ &\rho\,(v_t + (v\cdot\nabla)v) + \nabla p + \rho\nabla\Phi = 0 \\ &\rho\,(\varepsilon_t + (v\cdot\nabla)\varepsilon) + p\nabla\cdot v = \nabla\cdot(\nu\nabla T) \\ &\Delta\Phi = \rho, \quad \nu = \nu_0 T^n/\rho^m, \quad p = A\rho^\gamma \\ &\frac{(z+1)k_B}{\mu}T = \frac{p}{\rho} = (\gamma-1)\varepsilon \qquad \text{in } \mathbf{R}^3\times(0,T), \end{aligned} \tag{2.58}$$

where n, m, v_0 are physical constants and ε, T, k_B, μ, γ, and Z are the specific internal energy, the temperature, the Boltzmann constant, the mean atomic mas, the specific heat ratio, and the ionization state, respectively. Similar to (2.3), there is a type (II) self-similar blowup, see.[208] To take the first mathematical approach, we assume the case of $v_0 = 0$. Then, system (2.58) is reduced to

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho v) = 0 \\ &e_t + \nabla \cdot (ev) = 0 \\ &(\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p + \rho \nabla \Phi = 0 \\ &\Delta \Phi = \rho, \quad p = (\gamma - 1) e^{\gamma - 1} \rho \qquad \text{in } \mathbf{R}^3 \times (0, T), \end{aligned} \tag{2.59}$$

using $\varepsilon = e^{\gamma-1}$, and, therefore,

$$\frac{dM}{dt} = \frac{dE}{dt} = \frac{dH}{dt} = 0$$

holds for

$$\begin{aligned} M &= \int_{\mathbf{R}^3} \rho \\ E &= \int_{\mathbf{R}^3} e \\ H &= \int_{\mathbf{R}^3} \frac{\rho}{2} |v|^2 + \frac{p}{\gamma - 1} \, dx - \frac{1}{2} \langle \Gamma * \rho, \rho \rangle. \end{aligned}$$

We define the variational equilibrium by the critical state of the variational functional

$$H(\rho, e) = \int_{\mathbf{R}^3} \rho e^{\gamma - 1} - \frac{1}{2} \langle \Gamma * \rho, \rho \rangle$$

constrained by

$$\begin{aligned} e \geq 0, \ \int_{\mathbf{R}^3} e = E \\ \rho \geq 0, \ \int_{\mathbf{R}^3} \rho = M. \end{aligned}$$

First, $H_e = 0$ with $\int_{\mathbf{R}^3} e = E$ implies

$$\begin{aligned} &\rho e^{\gamma - 2} = \text{constant} \qquad \text{in } \mathbf{R}^3 \\ &\int_{\mathbf{R}^3} e = E \end{aligned}$$

and hence

$$e = \frac{E \rho^{\frac{1}{2-\gamma}}}{\int_{\mathbf{R}^3} \rho^{\frac{2}{2-\gamma}}}.$$

Next, $H_\rho = 0$ with $\int_{\mathbf{R}^3} \rho = M$ implies

$$\begin{aligned} &\Gamma * \rho = e^{\gamma - 1} + \text{constant} \qquad \text{in } \mathbf{R}^3 \\ &\int_{\mathbf{R}^3} \rho = M. \end{aligned}$$

Thus, it holds that

$$\Gamma * \rho = \frac{E^{\gamma-1}\rho^{\frac{\gamma-1}{2-\gamma}}}{\left(\int_{\mathbf{R}^3}\rho^{\frac{1}{2-\gamma}}\right)^{\gamma-1}} + \text{constant} \qquad \text{in } \mathbf{R}^3 \tag{2.60}$$

$$\rho \geq 0, \ \int_{\mathbf{R}^3}\rho = M. \tag{2.61}$$

To define the equilibrium on the bounded domain, we replace $\Gamma*$ by $(-\Delta_D)^{-1}$. Problem (2.61) has a similar form to (2.12) except for the non-local term, and we obtain $\rho = v_+^{\frac{2-\gamma}{\gamma-1}}$ similarly. It follows that

$$-\Delta v = E^{-\gamma+1}\left(\int_\Omega v_+^{\frac{1}{\gamma-1}}\right)^{\gamma-1} v_+^{\frac{2-\gamma}{\gamma-1}} \quad \text{in } \Omega, \qquad v = \text{constant} \quad \text{on } \partial\Omega$$

$$\int_\Omega v_+^{\frac{2-\gamma}{\gamma-1}} = M \tag{2.62}$$

Putting $\gamma = 1 + \frac{1}{q}$ and $v = cw$ with a constant $c > 0$, we see that (2.62) has the dimensionless form comparable to (2.26),

$$-\Delta w = w_+^{q-1} \quad \text{in } \Omega, \qquad w = \text{constant on } \partial\Omega$$

$$\int_\Omega w_+^q = \lambda \tag{2.63}$$

unless $q = 2$, that is $\gamma = 3/2$, where $\lambda = M^{\frac{q-1}{q-2}}E^{-q+2}$. Then, the variational functionals are defined by

$$J(w) = \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{q}\int_\Omega w_+^q + \lambda w_\Gamma, \qquad w \in H_c^1(\Omega)$$

$$J^*(u) = \frac{1}{\gamma'}\int_\Omega u^{\gamma'} - \frac{1}{2}\left\langle(-\Delta_D)^{-1}u, u\right\rangle, \quad u \geq 0, \ \int_\Omega u = \lambda,$$

where $\gamma' = 1 + \frac{1}{q-1}$. Then, we can detect the critical exponent for mass quantization to (2.63), $q = \frac{2n}{n-2}$, see.[306] A close result is known for

$$-\Delta v = |v|^{\frac{4}{n-2}}v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

$$\int_\Omega |v|^{\frac{2n}{n-2}} = \lambda$$

by.[294]

2.1.6. Summary

We have observed semi-duality and the quantized blowup mechanism in the fundamental equations of fluid dynamics.

(1) There is a duality between self-gravitating fluid and plasma confinement.

(2) The Nehari principle is involed by Temam's variational formulation of the plasma confinement, while there is a semi-correspondence of Morse indices between Berestycki-Brezis' formulation. Then, several stabilities of the linearized stable equilibrium of the Euler-Poisson equation are obtained by this semi-duality.
(3) A quantized blowup mechanism is observed in the non-stationary Euler-Poisson equation of the excellent radial pressure case. There are, however, several other possibilities of the blowup mechanism due to the lack of well-posedness of the problem or that of blowup criteria.

2.2. Particle Kinetics

The equilibrium statistical mechanics provides the derivation of the mean field limit of many self-interacting particles in the equilibrium state. In this section, first, we describe the theory of statistical mechanics in accordance with the vortex system and then the method of Kramers-Moyal and the kinetic theory. The Smoluchowski-Poisson equation (1.8) is then formulated again with its relatives in higher-dimension.

2.2.1. Vortex

If the vector field

$$v = \begin{pmatrix} v^1(x) \\ v^2(x) \\ v^3(x) \end{pmatrix} \in \mathbf{R}^3, \qquad x = (x_1, x_2, x_3) \in \mathbf{R}^3$$

denotes the velocity of the fluid, its rotation,

$$\nabla \times v = \begin{pmatrix} \frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1} \\ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \end{pmatrix},$$

stands for the rigid-like movement of this fluid, and is called the *vorticity*.
To understand this formula, let $\mathscr{O} \subset \mathbf{R}^3$ be a rigid body moving with a fixed point, denoted by the origin. We take an ortho-normal basis moving with $\mathscr{O}$, and let $\{i(t), j(t), k(t)\}$ be its position at the time t. Since $\mathscr{O}$ is rigid, it holds that

$$\begin{aligned} i(t)\cdot j(t) = j(t)\cdot k(t) = k(t)\cdot i(t) = 0 \\ i(t)\cdot i(t) = j(t)\cdot j(t) = k(t)\cdot k(t) = 1 \end{aligned}$$

and, therefore,

$$\begin{aligned} & i'\cdot j + i\cdot j' = j'\cdot k + j\cdot k' = k'\cdot i + k\cdot i' = 0 \\ & i'\cdot i = j'\cdot j = k'\cdot k = 0. \end{aligned} \tag{2.64}$$

Describing $\{i'(t), j'(t), k'(t)\}$ by $\{i(t), j(t), k(t)\}$,

$$\begin{aligned} i' &= c_{11}i + c_{12}j + c_{13}k \\ j' &= c_{21}i + c_{22}j + c_{23}k \\ k' &= c_{31}i + c_{32}j + c_{33}k, \end{aligned} \tag{2.65}$$

we obtain

$$\begin{aligned} &c_{23} + c_{32} = c_{31} + c_{13} = c_{12} + c_{21} = 0 \\ &c_{11} = c_{22} = c_{33} = 0 \end{aligned}$$

by (2.64). Thus (2.65) is reduced to

$$\begin{aligned} i' &= \quad c_3 j \quad -c_2 k \\ j' &= -c_3 i \quad\quad +c_1 k \\ k' &= c_2 i \quad -c_1 j \end{aligned} \tag{2.66}$$

using $c_1 = c_{23} = -c_{32}$, $c_2 = c_{31} = -c_{13}$, $c_3 = c_{12} = -c_{21}$, and the vector

$$\omega = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

is called the *angular velocity*.
If $x(t)$ denotes the position vector at the time t moving with $\mathscr{O}$, then it follows that

$$x(t) = x_1(0)i(t) + x_2(0)j(t) + x_3(0)k(t),$$

where

$$x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}.$$

This formula implies

$$\begin{aligned} x'(0) &= x_1(0)i'(0) + x_2(0)j'(0) + x_3(0)k'(0) \\ &= \begin{pmatrix} c_2(0)x_3(0) - c_3(0)x_2(0) \\ c_3(0)x_1(0) - c_1(0)x_3(0) \\ c_1(0)x_2(0) - c_2(0)x_1(0) \end{pmatrix} = \omega \times x|_{t=0}, \end{aligned}$$

and, therefore,

$$v = \frac{dx}{dt} = \omega \times x \tag{2.67}$$

at $t = 0$.
Relation (2.67) is valid to each t, and, therefore, this rigid body is infinitesimally rotating along ω with the speed $|\omega|$ in the direction where ω, x, and v form a right-handed coordinate system. It implies also

$$\nabla \times v = 2\omega,$$

and thus, the rotation $\nabla \times v$ of the velocity v is twice of the angular momentum if the fluid moves like a rigid body. This relation is what we have described at the begining of this paragraph.

Euler Equation

The velocity $v = v(x,t)$ of non-viscous incompressible fluid is subject to the Euler equation of motion and the equation of continuity described by (2.1) and (2.2), respectively, where ρ is a constant. This equation comprises of

$$\begin{aligned} & v_t + (v \cdot \nabla) v = -\nabla p \\ & \nabla \cdot v = 0 \qquad \text{in } \mathbf{R}^3 \times (0,T) \end{aligned} \tag{2.68}$$

when the outer force F is zero, where p denotes the pressure. If the vorticity of v is denoted by $\omega = \nabla \times v$, then (2.74) implies

$$\begin{aligned} & \omega_t + (v \cdot \nabla)\omega = (\omega \cdot \nabla) v \\ & \nabla \cdot v = 0 \qquad \text{in } \mathbf{R}^3 \times (0,T). \end{aligned} \tag{2.69}$$

Here, we consider the two-dimensional case, described by

$$v^3 = 0, \quad v^1 = v^1(x_1,x_2,t), \quad v^2 = v^2(x_1,x_2,t).$$

Then it holds that

$$\omega = \nabla \times v = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \end{pmatrix},$$

and, therefore, using the two-dimensional scalar field $\frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$, still denoted by ω, we obtain

$$\omega_t + (v \cdot \nabla)\omega = 0, \quad \nabla \cdot v = 0 \qquad \text{in } \mathbf{R}^2 \times (0,T),$$

or, equivalently,

$$\omega_t + \nabla \cdot (v\omega) = 0, \quad \nabla \cdot v = 0 \qquad \text{in } \mathbf{R}^2 \times (0,T). \tag{2.70}$$

Regarding

$$\nabla \cdot v = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} = 0,$$

we define the scalar field $\psi = \psi(x_1,x_2)$, called the *stream function*, by

$$v^1 = \frac{\partial \psi}{\partial x_2}, \qquad v^2 = -\frac{\partial \psi}{\partial x_1}.$$

This relation means $v = \nabla^\perp \psi$ for

$$\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix},$$

and then it holds that

$$\omega = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} = -\Delta\psi. \tag{2.71}$$

Thus (2.70) is reduced to

$$\omega_t + \nabla\cdot(\omega\nabla^\perp\psi) = 0, \quad -\Delta\psi = \omega \qquad \text{in } \mathbf{R}^2\times(0,T). \tag{2.72}$$

If $\Omega\subset\mathbf{R}^n$, $n=2,3$, is a bounded domain with smooth boundary $\partial\Omega$, the Euler equation (2.68) is taken place of

$$\begin{aligned} & v_t + (v\cdot\nabla)v = -\nabla p \\ & \nabla\cdot v = 0 && \text{in } \Omega\times(0,T) \\ & \nu\cdot v = 0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{2.73}$$

In the case that $\Omega\subset\mathbf{R}^2$ is simply-connected, we can define the stream function $\psi=\psi(x,t)$ by

$$v = \nabla^\perp\psi$$

using the solenoidal condition $\nabla\cdot v=0$. Then the boundary condition $\nu\cdot v=0$ of (2.73) is recovered by

$$\psi = \text{constant} \qquad\qquad \text{on } \partial\Omega\times(0,T).$$

Thus (2.73) is reduced to

$$\begin{aligned} & \omega_t + \nabla\cdot(\omega\nabla^\perp\psi) = 0 \\ & \psi(\cdot,t) = \int_\Omega G(\cdot,x')\omega(x',t)dx' \qquad \text{in } \Omega\times(0,T) \end{aligned} \tag{2.74}$$

similarly to (2.70), where $G=G(x,x')$ denotes the Green's function for $-\Delta_D$.

Vortex Equation

The problem (2.74) has a similar structure to the simplified system of chemotaxis, that is (1.125) and takes the weak form

$$\frac{d}{dt}\int_\Omega \varphi\omega = \frac{1}{2}\iint_{\Omega\times\Omega} \rho_\varphi\cdot\omega\otimes\omega \tag{2.75}$$

for $\varphi\in C^1(\overline{\Omega})$ with $\varphi|_{\partial\Omega}=0$, where

$$\begin{aligned} & \omega\otimes\omega = (\omega\otimes\omega)(x,x',t) = \omega(x,t)\omega(x',t) \\ & \rho_\varphi = \rho_\varphi(x,x') = \nabla_x^\perp G(x,x')\cdot\nabla\varphi(x) + \nabla_{x'}^\perp G(x.x')\cdot\nabla\varphi(x'). \end{aligned}$$

Using

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|},$$

we obtain

$$G(x,x') = \Gamma(x-x') + K(x,x')$$
$$K \in C^{2,\theta}(\overline{\Omega}\times\Omega \cup \Omega\times\overline{\Omega}),$$

and, therefore,

$$\begin{aligned}
\iint_{\Omega\times\Omega} \rho_\varphi \omega\otimes\omega &= \iint_{\Omega\times\Omega} \nabla^\perp \Gamma(x-x')\cdot\left[\nabla\varphi(x)-\nabla\varphi(x')\right]\omega\otimes\omega \\
&+ \iint_{\Omega\times\Omega}\left[\nabla_x^\perp K(x,x')\cdot\nabla\varphi(x) + \nabla_{x'}^\perp K(x,x')\cdot\nabla\varphi(x')\right]\omega\otimes\omega \\
&= \lim_{\varepsilon\downarrow 0}\iint_{\Omega\times\Omega\setminus\{|x-x'|<\varepsilon\}} \nabla^\perp\Gamma(x-x')\cdot\left[\nabla\varphi(x)-\nabla\varphi(x')\right]\omega\otimes\omega \\
&+ \iint_{\Omega\times\Omega}\left[\nabla_x^\perp K(x,x')\cdot\nabla\varphi(x) + \nabla_{x'}^\perp K(x,x')\cdot\nabla\varphi(x')\right]\omega\otimes\omega,
\end{aligned} \tag{2.76}$$

provided that $\omega(\cdot,t)\in L^1(\Omega)$ and supp $\varphi\subset\Omega$, see.[76,192,272,304]
Now, we apply (2.75)-(2.76) to

$$\omega(dx,t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx)$$

with $\alpha_i\in\mathbf{R}$ and $x_i(t)\in\Omega$ ($i=1,\ldots,N$), and obtain the *vortex equation*

$$\frac{dx_i}{dt} = \frac{1}{2}\alpha_i\nabla^\perp R(x_i) + \sum_{j=1,j\neq i}^N \alpha_i\alpha_j \nabla_x^\perp G(x_i,x_j) \tag{2.77}$$

for $i=1,\ldots,N$, where $R(x)=K(x,x)$. This equation is formulated by the Hamilton system

$$\begin{aligned}
\alpha_i\frac{dx_{i1}}{dt} &= \frac{\partial H}{\partial x_{i2}} \\
\alpha_i\frac{dx_{i2}}{dt} &= -\frac{\partial H}{\partial x_{i1}},
\end{aligned} \tag{2.78}$$

where $x_i=(x_{i1},x_{i2})$ and

$$\begin{aligned}
H = H(x_1,\ldots,x_N) &= \frac{1}{2}\sum_i \alpha_i^2 R(x_i) + \sum_{i<j}\alpha_i\alpha_j G(x_i,x_j) \\
&= \frac{1}{2}\sum_{i=1}^N\sum_{j=1,j\neq i}^N \alpha_i\alpha_j G(x_i,x_j) + \frac{1}{2}\sum_{i=1}^N \alpha_i^2 R(x_i).
\end{aligned} \tag{2.79}$$

2.2.2. Boltzmann Relation

If the Hamiltonian is given, then we can define the probability densities describing macroscopic state of large number of particles. This formulation is based on the principle of equal *a priori* probabilities, and in this paragraph we derive Boltzmann's relation by this principle.

First, Boyle-Charles' law holds in the ideal gas, and hence it follows that

$$pV = RT,$$

where p, V, T, and R are the pressure, the volume, the temperature, and the gas constant, respectively. We have also

$$U = c_v T,$$

where c_v is the specific heat and U is the inner energy. If the process is reversible, it holds that

$$dS = \frac{d'Q}{T} = \frac{dU + pdV}{T} = c_v \frac{dT}{T} + R\frac{dV}{V}.$$

We obtain, therefore,

$$S(T,V) = c_v \log T + R \log V,$$

up to the additive constant. This formula implies

$$\Delta S = R \log \frac{V_2}{V_1} > 0$$

under the dilation of the gas with $\Delta T = 0$, where

$$\Delta V = V_2 - V_1 > 0.$$

Let us consider the boxrooms of which volumes are equally v_0. Then, their numbers are $M_1 = V_1/v_0$ and $M_2 = V_2/v_0$ before and after the dilation, respectively. The numbers of divisions of moleculars into these boxrooms are

$$W_1 = \frac{M_1^N}{N!} = \frac{1}{N!}\left(\frac{V_1}{v_0}\right)^N$$
$$W_2 = \frac{M_2^N}{N!} = \frac{1}{N!}\left(\frac{V_2}{v_0}\right)^N,$$

where $N = N_A$ is the Avogadro constant. Thus we obtain

$$\Delta S = \frac{R}{N} \log \frac{W_1}{W_2},$$

which results in

$$S = k \log W, \tag{2.80}$$

where $k = R/N_A$ is the Boltzmann constant. This W is called the *thermal weight factor*. It thus indicates the number of microscopic states admitted by the prescribed macroscopic state.

This macroscopic state is divided, furthermore, into many *mezoscopic* states, labeled by $i = 1, 2, \cdots$, and then we obtain

$$W = \prod_i \frac{g_i^{n_i}}{n_i!},$$

where g_i and n_i denote the numbers of microscopic states and of particles, respectively, taking the mezoscopic state i. Here, each microscopic state takes $f_i = n_i/g_i$ particles from the principle of equal *a priori* probabilities. Using Stirling's formula

$$\log n! \sim n(\log n - 1), \qquad n \to \infty,$$

we obtain

$$\begin{aligned} S &= k\sum_i (n_i \log g_i - \log n_i!) \\ &\sim k\sum_i (n_i \log g_i - n_i(\log n_i - 1)) \\ &= k\sum_i (f_i g_i \log g_i - f_i g_i(\log f_i + \log g_i - 1)) \\ &= -k\sum_i g_i f_i(\log f_i - 1). \end{aligned}$$

If $f = f(x,p)$ indicates the mean density of particles with the phase variable (x,p), then we replace g_i by $\Delta x_i \Delta p_i / h$ using microcanonical statistics and uncertainty principle, where h denotes the Planck constant. This formulation implies

$$S = -\frac{k}{h^{3N}} \iint f(\log f - 1)dxdp$$

for $dxdp = dp_1 dx_1 \dots dp_N dx_N$.

2.2.3. Ensemble

Ensemble is a fundamental concept of statistical mechanics, indicating the equivalent class of the microscopic states that have the same macroscopic profile. Then, the mission of the equilibrium statistical mechanics is to define their probability measure in the equilibrium using the principle of equal *a priori* probabilities. There are micro-canonical, canonical, and grand-canonical ensembles, associated with the materially and kinetically closed, materially closed, and open systems, respectively. The micro-canonicial ensemble is prescribed by N, V, and E, the canoical ensemble is prescribed by N, V, and T, and the grand-canonical ensemble is prescribed by V, T, μ, respectively, where N, V, E, T, and μ stand for the number of particles, the volume, the energy, the temperature, and the chemical potential, respectively.

Micro-Canonical Statistical Mechanics

First, micro-canonical statistical mechanics is concerned with the kinetically and materially closed system. Such a system is described by the Hamilton system

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \qquad 1 \le i \le N, \end{aligned} \tag{2.81}$$

where $(q_1, \dots, q_N) \in \mathbf{R}^{3N}$ and $(p_1, \dots, p_N) \in \mathbf{R}^{3N}$ denote the *general position* and the *general momentum* of self-interacting particles, respectively, and

$$H = H(q_1, \dots, q_N, p_1, \dots, p_N)$$

is the *Hamiltonian* (§2.3.1).

We obtain

$$\frac{d}{dt}H(q(t),p(t)) = 0,$$

and, therefore, the total energy H is a constant, denoted by E, while each point in the phase space $(p_1,\ldots,p_N,q_1,\ldots,q_N) \in \Gamma = \mathbf{R}^{6N}$ is regarded as a microscopic state. Thus the micro-canninical ensemble is the equivalent class of Γ with the equivalent relation defined by this total energy. Here, the co-area formula guarantees

$$dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|},$$

where $dx = dq_1 \ldots dq_N dp_1 \ldots dp_N$, and $d\Sigma(E)$ is the surface element on

$$\{x \in \Gamma \mid H(x) = E\},$$

and, therefore, the probability measure of the micro-canonical ensemble is defined by

$$\mu^{E,N}(dx) = \frac{1}{\Omega(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}$$
$$\Omega(E) = \int_{\{H=E\}} \frac{d\Sigma(E)}{|\nabla H|} \tag{2.82}$$

for each $E \in \mathbf{R}$, from the principle of equal *a priori* probabilities.
Thus

$$\langle f \rangle = \langle f \rangle_{E,N} = \int_{\{H=E\}} f(x)\mu^{E,N}(dx)$$

indicates the *phase mean* of the physical quantity f over $\{H = E\}$. From the *ergodic hypothesis*, this value $\langle f \rangle$ is assumed to be equal to the time mean along the orbit,

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(T_t x)dt,$$

for $d\Sigma(E)$-a.e. in $x \in \{H = E\}$, where $\{T_t\}$ denotes the semi-group associated with (2.81). This value $\overline{f}(x)$ is regarded as an observable macroscopic datum, and the ergodic hypothesis asserts the practical efficiency of the equilibrium statistical mechanics.

Canonical Statistical Mechanics

Canonical statistical mechanics is concerned with the materially closed system with prescribed temperature. To assign the probabilities of such ensembles, first, we take a kinetically and materially closed system of n-particles provided with the Hamiltonian H. We use the coordinate

$$(x_1,\ldots,x_{2n}) \in \Gamma = \mathbf{R}^{6n}$$

instead of $(p_1,\ldots,p_n,q_1,\ldots q_n)$ for simplicity, and decompose this phase space into $\Gamma = \Gamma_1 \oplus \Gamma_2$ by $(x_1,\ldots,x_{2N}) \in \mathbf{R}^{6N}$ and $(x_{2N+1},\ldots,x_{2n}) \in \mathbf{R}^{6n-6N}$. Prescribing the total energy by $H = E$, we obtain the sets of microscopic states constrained by $H = E$ in Γ_1 and Γ_2, denoted by G_1 and G_2, respectively.

Let $\mu_1(E_1)$ be the probability of the microscopic states of G_1 of which energies are E_1. Then, from the principle of equal *a priori* probabilities it follows that

$$\mu_1(E_1) = \frac{\Omega_1(E_1)\Omega_2(E-E_1)}{\Omega(E)}, \tag{2.83}$$

where $\Omega(E)$, $\Omega_1(E_1)$, and $\Omega_2(E_2)$ denote the numbers of the microscopic states of Γ, Γ_1, and Γ_2 with the total energies E, E_1, and E_2, respectively.
We assume $N \ll n$, and regard G_1 and G_2 as a materially closed equilibrium system and a heat bath, respectively, where the temperature T is prescribed. In this case, $E_1 = E_1^*$ attains the maximum of $\mu_1(E_1)$, and it holds that

$$0 = \frac{\partial\Omega_1(E_1)}{\partial E_1}\cdot\frac{\Omega_2(E-E_1)}{\Omega(E)} + \frac{\Omega_1(E_1)}{\Omega(E)}\cdot\frac{\partial\Omega_2(E-E_1)}{\partial E_1}$$

by (2.83), and hence

$$\frac{\partial}{\partial E_1}\log\Omega_1(E_1) = \frac{\partial}{\partial E_2}\log\Omega_2(E_2),$$

where $E_2 = E - E_1$. Here, we confirm that $E_1 = E_1^*$ is prescribed and that the temperatures of G_1 and G_2 are equal because they are in the equilibrium. Thus

$$\frac{\partial}{\partial E_2}\log\Omega_2(E_2)$$

is a constant determined by T.
Regarding (2.80) and (1.234), now we infer

$$S_2 = k\log\Omega_2$$
$$\left(\frac{\partial S_2}{\partial E}\right)_V = \frac{1}{T},$$

and conclude

$$\frac{\partial}{\partial E_2}\log\Omega_2(E_2) = \beta$$
$$\Omega_2(E_2) = \text{constant}\times e^{\beta E_2},$$

where $\beta = 1/(kT)$ is called the *inverse temperature*. Still $E_1 = E_1^*$ is prescribed, and this guarantees that $\mu_1(E_1)$ is proportional to $e^{-\beta E_1}$ by (2.83). Writing $\mu^{\beta,N}(dx)$, H, and Γ for $\mu_1(E_1)$, E_1, and Γ_1, respectively, thus we obtain the probability measure of the canonical ensemble, called the *Gibbs mesure*,

$$\mu^{\beta,N}(dx) = \frac{e^{-\beta H}dx}{Z(\beta,N)}$$
$$Z(\beta,N) = \int_\Gamma e^{-\beta H}dx. \tag{2.84}$$

If we define the entropy, the inner energy, and the free energy of the macrostate by

$$S = -k\int_\Gamma \mu(\log\mu - 1)dx$$
$$E = \int_\Gamma H\mu(dx),$$

and $F = -TS + E$, respectively, then the above particle density $\mu = \mu^{\beta,N}(dx)$ is the minimizer of $F = F(\mu)$ defined for the probability measure $\mu = \mu(dx)$ on Γ. In fact, Euler-Lagrange equation for this variational problem is described by

$$\log \mu + \beta H = \text{constant},$$

and then the minimizer $\mu = \mu^{\beta,N}(dx)$ is defined by (2.84), using $\mu(\Gamma) = 1$.
We note that in the case of model (B) equation derived from Helmholtz' free energy, the Hamiltonian describing inner energy is associated with the particle density through the self-interaction potential, and then (2.84) induces an elliptic eigenvalue problem involving exponential nonlinearity and non-local term, see §1.1.4.

Grand-Canonical Statistical Mechanics

Grand-canonical statistical mechanics is concerned with the open system with prescribed temperature and pressure. To assign the probabilities of such ensembles, we follow the argument of canoncial statistical mechanics. Thus we take a kinetically and materially closed system of n-particles provided with the Hamiltonian H, use the coordinate $(x_1, \dots, x_{2n}) \in \Gamma = \mathbf{R}^{6n}$ to indicate these particles, decompose the phase space into $\Gamma = \Gamma_1 \oplus \Gamma_2$ by $(x_1, \dots, x_{2N}) \in \mathbf{R}^{6N}$ and $(x_{2N+1}, \dots, x_{2n}) \in \mathbf{R}^{6n-6N}$, prescribe the total energy by $H = E$, and obtain the microscopic states in Γ_1 and Γ_2, denoted by G_1 and G_2, respectively.
Then, we take the probability $\mu_1(E_1, n_1)$ of the microscopic states of G_1 of which energies and particle numbers are E_1 and n_1, respectively. From the principle of equal *a priori* probabilities, now it follows that

$$\mu_1(E_1, n_1) = \frac{\Omega_1(E_1, n_1)\Omega_2(E - E_1, n - n_1)}{\Omega(E, n)}, \tag{2.85}$$

where $\Omega(E, n)$, $\Omega_1(E_1, n_1)$, and $\Omega_2(E_2, n_2)$ denote the total quantities of microscopic states of Γ, Γ_1, and Γ_2 with the total energies E, E_1, and E_2, and the particle numbers n, n_1, and n_2, respectively.
We assume $N \ll n$, and regard G_1 and G_2 as an open equilibrium system and a particle bath, respectively, where the temperature T and the pressure p are prescribed. In this case, $(E_1, n_1) = (E_1^*, n_1^*)$ attains the maximum of $\mu_1(E_1, n_1)$, and, therefore, (2.85) implies

$$\begin{aligned} \frac{\frac{\partial}{\partial E_1}\Omega_1(E_1, n_1)}{\Omega_1(E_1, n_1)} &= \frac{\frac{\partial}{\partial E_2}\Omega_2(E_2, n_2)}{\Omega_2(E_2, n_2)} \\ \frac{\frac{\partial}{\partial n_1}\Omega_1(E_1, n_1)}{\Omega_1(E_1, n_1)} &= \frac{\frac{\partial}{\partial n_2}\Omega_2(E_2, n_2)}{\Omega_2(E_2, n_2)}, \end{aligned} \tag{2.86}$$

where $E_2 = E - E_1$ and $n_2 = n - n_1$.
The first relation of (2.86) implies

$$\frac{\partial}{\partial E_2} \log \Omega_2(E_2, n_2) = \beta$$

similarly. The chemical potential, on the other hand, is defined by

$$\mu = \left(\frac{\partial G}{\partial n}\right)_{T,p} = -T\left(\frac{\partial S}{\partial n}\right)_{E,V}$$

from $G = H - TS$ and $H = E + pV$, and, therefore, it holds that

$$\frac{\partial}{\partial n_2}\log\Omega_2(E_2,n_2) = -\zeta$$

for $\zeta = \beta\mu$ by the second relation of (2.86). This property implies

$$\mu_1(E_1,n_1) = \text{constant} \times e^{\zeta n_1 - \beta E_1},$$

and, consequently, the probability measure of the grand-canonical ensemble is defined by

$$\mu^{\beta,\zeta}(n,dx) = \frac{e^{\zeta n - \beta H}dx}{\Xi(\beta,\zeta)}$$
$$\Xi(\beta,\zeta) = \int_\Gamma e^{\zeta n - \beta H} d\Gamma.$$

If there are a variety of particles, then it holds that

$$\mu^{\beta,\zeta}(n_k,dx) = \frac{e^{\zeta n_k - \beta H}dx}{\Xi(\beta,\zeta)}, \qquad k = 1,2,\ldots$$
$$\Xi(\beta,\zeta) = \sum_k \int_{\Gamma_k} e^{\zeta n_k - \beta H} d\Gamma_k,$$

where $\Gamma_k = \mathbf{R}^{6k}$ and $d\Gamma_k$ indicates the volume element.

2.2.4. Turbulence

One aspect of turbulence is illsustrated as a status of the fluid provided with a large number of vortex points. Two-dimensional vortices macroscopically stable for a log time period are observed often. This phenomenon is mentioned in the context of ordered structure, and motivated by this observation, Onsager[241] initiated the equilibrium statistical mechanics of many vortex points. More precisely, two-dimensional vortex points of the perfect fluid are described by the *Hamilton system* (2.78), whereby the above mentioned theory of equilibrium statistical mechanics is applied. Its physical validity, on the other hand, is examined in connection with the ergodic hypothesis.[84]

In this case, the phase space Γ is replaced by Ω^n and the probability measure of the ensemble is denoted by

$$\mu^n = \mu^n(dx_1,\ldots,dx_n),$$

where $n \gg 1$ is the number of vortex points and

$$X^n = (x_1,\ldots,x_n) \in \Omega^n.$$

We can define $\rho^n_{1,i}(x_i)$ by

$$\rho^n_{1,i}(x_i)dx_i = \int_{\Omega^{n-1}} \mu^n(dx_1 \dots dx_{i-1}dx_{i+1}dx_n),$$

which is independent of $i = 1, \dots, n$ from the principle equal *a priori* probabilities. This $\rho^n_{1,i}(x_i)$ is denoted by $\rho^n_1(x)$, and is called the one-point reduced probability density function, or (reduced) "pdf" in short. Similarly, the k-point reduced probability density function is defined by

$$\rho^n_k(x_1, \dots, x_k)dx_1 \dots dx_k = \int_{\Omega^{n-k}} \mu^n(dx_{k+1}, \dots, dx_n). \tag{2.87}$$

The stationary n-vortex points $\omega = \omega_n(x)$ with the intensities equal to the same value $\alpha > 0$ is described by

$$\omega_n(x)dx = \sum_{i=1}^{n} \alpha\delta_{x_i}(dx)$$

and, therefore, its phase mean is defined by

$$\begin{aligned} \langle\omega_n(x)\rangle &= \sum_{i=1}^{n} \int_{\Omega^n} \alpha\delta(x_i - x)\mu^n(dx_1 \dots dx_n) \\ &= n\alpha\rho^n_1(x), \end{aligned}$$

using the one-point pdf $\rho^n_1 = \rho^n_1(x)$.
In the *high-energy limit*, we assume $n \to \infty$ with $\alpha n = 1$, $\alpha^2 n^2 \tilde{E} = E$, and $\alpha^2 n\tilde{\beta} = \beta$, where $\tilde{E}$ and $\tilde{\beta}$ are the energy and the inverse temperature of the n-vortex system, respectively, and E and β are constants. In this case, the *mean field* $\rho = \rho(x)$ is defined by

$$\lim_{n\to\infty} \langle\omega_n(x)\rangle = \rho(x) = \lim_{n\to\infty} \rho^n_1(x), \tag{2.88}$$

which is expected to satisfy the *mean field equation*,

$$\begin{aligned} \rho &= \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \\ \psi &= \int_\Omega G(\cdot, x')\rho(x')dx' \qquad \text{in } \Omega. \end{aligned} \tag{2.89}$$

Thus two-dimensional stationary turbulence is provided with the dual variation between the "particle density" ρ and the "field" ψ.
If equation (2.89) admits a unique solution, conversely, then the limiting process (2.88) is justified in the sense of measure.[39] This property is actually the case of $\beta > -8\pi$. Furthermore, there is *propagation of chaos* indicated by

$$\rho^n_k \quad \rightharpoonup \quad \rho^{\otimes k}$$

in the sense of measure, where $\rho^n_k(x_1, \dots, x_k)$ is defined by (2.87) and

$$\rho^{\otimes k}(x_1, \dots, x_k) = \prod_{i=1}^{k} \rho(x_i).$$

In the general case, it holds that

$$\rho_k^n \rightharpoonup \int \rho^{\otimes k} \xi(d\rho)$$

passing to a subsequence with a probability measure $\xi(d\rho)$, and if ρ is in the support of ξ, then it is a minimizer of a free energy functional.[39,143]
Actually, these results are obtained in the context of equilibrium statistical mechanics using the Hamiltonian (2.79). Although the above β is called the inverse temperature, it is not associated with the physical temperature, and there is a possibility of negative β. In fact, using the micro-canonical probability mesure $\Omega(E)$ defined by (2.82), we obtain

$$\beta = \frac{\partial}{\partial E} \log \Omega(E)$$

by (2.80) and (1.234). We have, on the other hand,

$$\Theta(E) \equiv \int_{\{H<E\}} dx_1 \ldots dx_n = \int_{-\infty}^{E} \Omega(E')dE'$$

by the co-area formula and (2.82), and therefore,

$$\tilde{\beta} = \frac{\Theta''(E)}{\Theta'(E)}.$$

Since $\Theta(E)$ is a monotone increasing bounded function of $E \in \mathbf{R}$, it has a point of inflection, and $\tilde{\beta} < 0$ occurs for $E \gg 1$, see.[241] The other argument uses canonical statistical mechanics to confirm the case $\beta < 0$, see.[84]

Deterministic Intensity

The above described mean field equation is concerned with a uniform intensity, but vortices with non-uniform intensities can also arise. These intensities may be either deterministic or stochastic.
First, there are several physical and mathematical arguments to derive the stationary turbulence mean field equation for deterministic intensities.[39,40,159,170,258] Here, we apply the method of[159] concerning the signed vortices, and show the general form. In more detail, we use the micro-canonical statistical mechanics, assuming $n^1 n, \ldots, n^I n$-numbers of vortices with the intensities $\alpha^1 \alpha, \ldots, \alpha^I \alpha$, respectively, where $0 < n^i < 1$, $-1 \le \alpha^i \le 1$ $(i = 1, \ldots, I)$, and $\sum_{i=1}^I n^i = 1$.
First, we divide Ω into small cells $\{\Delta_k\}$ with the same volume $\Delta = |\Delta_k|$ $(k = 1, 2, \ldots)$. If N_k^i denotes the number of vortices of the intensity $\alpha^i \alpha$ contained in Δ_k, then the weight factor indicating the number of microscopic states is defined by

$$W = \prod_{i=1}^{I} \prod_k \frac{\Delta^{N_k^i}}{N_k^i!},$$

and the equilibrium of $\{\{N_1^1, N_2^1, \ldots\}, \{N_1^2, N_2^2, \ldots\}, \ldots, \{N_1^I, N_2^I, \ldots\}\}$ is described by the condition that W attains the maximum under the constraint of

$$\sum_k N_k^1 = n^1 n, \quad \ldots \quad , \sum_k N_k^I = n^I n$$
$$H(x_1, \ldots, x_n) = \tilde{E}. \tag{2.90}$$

Here, x_k is the center of Δ_k, $k = 1, 2, \ldots$, and

$$H(x_1, \ldots, x_n) = \frac{\alpha^2}{2} \sum_k \sum_{\ell \neq k} \left(\sum_{i=1}^I \alpha^i N_k^i \right) \left(\sum_{i=1}^I \alpha^i N_\ell^i \right) G(x_k, x_\ell) + \frac{\alpha^2}{2} \sum_k \left(\sum_{i=1}^I \alpha^i N_k^i \right)^2 R(x_k)$$

denotes the Hamiltonian defined by (2.79) and

$$\alpha \sum_i \alpha^i N_k^i = \alpha_k.$$

Thus it holds that (2.90) and

$$\frac{\partial}{\partial N_k^i} \log W - c^1 \frac{\partial}{\partial N_k^i} \sum_{k'} N_{k'}^1 - c^2 \frac{\partial}{\partial N_k^i} \sum_{k'} N_{k'}^2 - \ldots - c^I \frac{\partial}{\partial N_k^i} \sum_{k'} N_{k'}^I$$
$$- \tilde{\beta} \frac{\partial}{\partial N_k^i} \left[\frac{\alpha^2}{2} \sum_{k'} \sum_{\ell' \neq k'} \left(\sum_{i=1}^I \alpha^i N_{k'}^i \right) \left(\sum_{i=1}^I \alpha^i N_{\ell'}^i \right) G(x_{k'}, x_{\ell'}) + \frac{\alpha^2}{2} \sum_{k'} \left(\sum_{i=1}^I \alpha^i N_{k'}^i \right)^2 R(x_{k'}) \right] = 0 \tag{2.91}$$

for $k = 1, 2, \ldots$ and $i = 1, \ldots I$, where $c^1, \ldots, c^I$ and $\tilde{\beta}$ are the Lagrangian multiplier. Since

$$\log W = \log \prod_{i=1}^I \prod_k \frac{\Delta^{N_k^i}}{N_k^i!} = \sum_{i=1}^I \log \left[\prod_k \frac{\Delta^{N_k^i}}{N_k^i!} \right] \approx \sum_{i=1}^I \sum_k \left[N_k^i \log \Delta - N_k^i \log N_k^i + N_k^i \right]$$

it holds that

$$\frac{\partial}{\partial N_k^i} \log W = \log \Delta - \log N_k^i,$$

and, therefore, the last term of the left-hand side of (2.91) is equal to

$$
\begin{aligned}
&\tilde{\beta}\left[\frac{\alpha^2}{2}\sum_{\ell'\neq k}\alpha^i\left(\sum_{i=1}^{I}\alpha^i N_{\ell'}^i\right)G(x_k,x_{\ell'})\right.\\
&\left.+\frac{\alpha^2}{2}\sum_{k'\neq k}\left(\sum_{i=1}^{I}\alpha^i N_{k'}^i\right)\alpha^i G(x_{k'},x_k)+\alpha^2\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)\alpha^i R(x_k)\right]\\
&=\tilde{\beta}\alpha^2\alpha^i\left[\sum_{\ell\neq k}\left(\sum_{i=1}^{I}\alpha^i N_\ell^i\right)G(x_k,x_\ell)+\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)R(x_k)\right].
\end{aligned}
$$

Thus it holds that

$$
\begin{aligned}
&\log\Delta-\log N_k^i-c^i-\tilde{\beta}\alpha^2\alpha^i\\
&\cdot\left[\sum_{\ell\neq k}\left(\sum_{i=1}^{I}\alpha^i N_\ell^i\right)G(x_k,x_\ell)+\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)R(x_k)\right]=0,
\end{aligned}
\tag{2.92}
$$

and, therefore, we obtain

$$
\begin{aligned}
\log W &= \sum_{i=1}^{I}\sum_k\left(N_k^i\log\Delta-N_k^i\log N_k^i+N_k^i\right)\\
&=\sum_{i=1}^{I}\sum_k N_k^i\left[c^i+\tilde{\beta}\alpha^2\alpha^i\left[\sum_{\ell\neq k}\left(\sum_{i=1}^{I}\alpha^i N_\ell^i\right)G(x_k,x_\ell)\right.\right.\\
&\qquad\left.\left.+\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)R(x_k)\right]\right]+n\\
&=n+\sum_{i=1}^{I}\sum_k c^i N_k^i+\tilde{\beta}\alpha^2\left[\sum_{i=1}^{I}\sum_k\alpha^i N_k^i\right.\\
&\qquad\left.\cdot\left[\sum_{\ell\neq k}\left(\sum_{i=1}^{I}\alpha^i N_\ell^i\right)G(x_k,x_\ell)+\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)R(x_k)\right]\right]\\
&=n+\sum_{i=1}^{I}\sum_k c^i N_k^i+\tilde{\beta}\alpha^2\left[\sum_k\sum_{\ell\neq k}\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)\left(\sum_{i=1}^{I}\alpha^i N_\ell^i\right)\right.\\
&\qquad\left.\cdot G(x_k,x_\ell)+\sum_k\left(\sum_{i=1}^{I}\alpha^i N_k^i\right)^2 R(x_k)\right]\\
&=n+\sum_{i=1}^{I}c^i n_i n+2\tilde{\beta}\tilde{E}.
\end{aligned}
$$

This formula implies

$$
2\tilde{\beta}=\frac{\partial}{\partial\tilde{E}}\log W=\frac{1}{k}\frac{\partial S}{\partial\tilde{E}},
$$

and, therefore, $2\tilde{\beta}$ describes the inverse temperature of this system.

Next, we obtain

$$N_k^i = \Delta e^{-c^i} \exp\left(-\tilde{\beta}\alpha^2\alpha^i \left[\sum_{\ell\neq k}\left(\sum_{i=1}^I \alpha^i N_\ell^i\right) G(x_k, x_\ell) + \left(\sum_{i=1}^I \alpha^i N_k^i\right) R(x_k)\right]\right) \tag{2.93}$$

by (2.92), and assume the high-energy limit,

$$\sum_i \alpha^i \frac{N_k^i}{n} \quad \approx \quad \rho(x_k)\Delta. \tag{2.94}$$

More precisely, we deduce

$$\begin{aligned} nn^i &= \sum_k N_k^i \\ &= e^{-c^i}\sum_k \Delta \exp\left(-\tilde{\beta}\alpha^2\alpha^i\left[\sum_{\ell\neq k}\left(\sum_{i=1}^I \alpha^i N_\ell^i\right) G(x_k,x_\ell) + \left(\sum_{i=1}^I \alpha^i N_k^i\right) R(x_k)\right]\right) \\ &\approx e^{-c^i}\sum_k \Delta\cdot \exp\left(-\frac{\beta}{2}\alpha^i\left[\sum_{\ell\neq k}\rho(x_\ell)G(x_k,x_\ell) + \rho(x_k)R(x_k)\right]\Delta\right) \end{aligned}$$

from (2.93) and $2\tilde{\beta}\alpha^2 = \beta/n$, and introduce the "stream function"

$$\psi(x) = \int_\Omega \rho(x')G(x',x)dx' = \int_\Omega G(x,x')\rho(x')dx'.$$

Using the principal value, then we obtain

$$\left[\sum_{\ell\neq k}\rho(x_\ell)G(x_k,x_\ell) + \rho(x_k)R(x_k)\right]\Delta \quad \approx \quad \psi(x_k),$$

and hence

$$\begin{aligned} nn^i &\approx e^{-c^i}\sum_k \Delta \exp\left(-\frac{\beta}{2}\alpha^i\psi(x_k)\right) \\ &\approx e^{-c^i}\int_\Omega \exp\left(-\frac{\beta}{2}\alpha^i\psi(x)\right)dx. \end{aligned}$$

This relation implies

$$\frac{e^{-c^i}}{n} = \frac{n^i}{\int_\Omega \exp\left(-\frac{\beta}{2}\alpha^i\psi\right)}$$

and hence it holds that

$$
\begin{aligned}
\rho(x_k) &\approx \sum_i \alpha^i \frac{N_k^i}{n\Delta} \\
&= \sum_i \alpha^i n^i \frac{e^{-c^i}}{n} \exp\left(-\tilde{\beta}\alpha^2\alpha^i \left[\sum_{\ell\neq k}\left(\sum_{i=1}^I \alpha^i N_\ell^i\right) G(x_k,x_\ell) \right.\right. \\
&\quad \left.\left. + \left(\sum_{i=1}^I \alpha^i N_k^i\right) R(x_k)\right]\right) \\
&\approx \sum_i \frac{\alpha^i n^i}{\int_\Omega \exp\left(-\frac{\beta}{2}\alpha^i\psi\right)} \\
&\quad \cdot \exp\left(-\frac{\beta}{2}\alpha^i \left[\sum_{\ell\neq k} \rho(x_\ell)G(x_k,x_\ell) + \rho(x_k)R(x_k)\right]\Delta\right) \\
&\approx \sum_i \alpha^i n^i \frac{\exp\left(-\frac{\beta}{2}\alpha^i\psi(x_k)\right)}{\int_\Omega \exp\left(-\frac{\beta}{2}\alpha^i\psi\right)}
\end{aligned}
$$

by (2.94) and (2.93).
Thus using

$$
P(d\alpha) = \sum_{i=1}^I n^i \delta_{\alpha^i}(d\alpha),
$$

we obtain

$$
\begin{aligned}
\rho &= \int_{[-1,1]} \frac{\alpha e^{-\frac{\beta}{2}\alpha\psi}}{\int_\Omega e^{-\frac{\beta}{2}\alpha\psi}} P(d\alpha) \\
\psi &= \int_\Omega G(\cdot,x')\rho(x')dx'.
\end{aligned} \tag{2.95}
$$

This formula implies

$$
\log\rho_\alpha + \frac{\beta}{2}(-\Delta_D)^{-1}\rho = \text{constant} \tag{2.96}
$$

for

$$
\rho_\alpha = \frac{e^{-\frac{\beta}{2}\alpha\psi}}{\int_\Omega e^{-\frac{\beta}{2}\alpha\psi}}
$$

satisfying

$$
\begin{aligned}
&\rho_\alpha \geq 0, \quad \int_\Omega \rho_\alpha = 1 \\
&\rho = \int_{[-1,1]} \alpha\rho_\alpha P(d\alpha).
\end{aligned} \tag{2.97}
$$

Equation (2.96) has the variational function

$$
\begin{aligned}
I(\oplus\rho_\alpha) &= \int_{[-1,1]}\left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right]P(d\alpha) + \frac{\beta}{4}\left\langle(-\Delta_D)^{-1}\rho,\rho\right\rangle \\
&= \int_{[-1,1]}\left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right]P(d\alpha) \\
&\quad + \frac{\beta}{4}\iint_{[-1,1]\times[-1,1]} \alpha\alpha'\left((-\Delta_D)^{-1/2}\rho_\alpha,(-\Delta_D)^{-1/2}\rho_{\alpha'}\right)P\otimes P(d\alpha d\alpha')
\end{aligned}
$$

constrained by (2.97). Its boundedness is studied when $P(d\alpha)$ is discrete, see.[62,284] We obtain, on the other hand,

$$
-\Delta\psi = \int_{[-1,1]} \frac{\alpha e^{-\frac{\beta}{2}\alpha\psi}}{\int_\Omega e^{-\frac{\beta}{2}\alpha\psi}} P(d\alpha) \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega \tag{2.98}
$$

by (2.95), which has the variational function

$$
J(\psi) = \frac{1}{2}\|\nabla\psi\|_2^2 + \frac{2}{\beta}\int_{[-1,1]}\log\left(\int_\Omega e^{-\frac{\beta}{2}\alpha\psi}\right)P(d\alpha) + \frac{2}{\beta}
$$

defined for $\psi \in H_0^1(\Omega)$. For the duality of these variations to confirm, we define the Lagrangian

$$
L(\oplus\rho_\alpha,\psi) = -\frac{2}{\beta}\int_{[-1,1]}\left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right]P(d\alpha) + \frac{1}{2}\|\nabla\psi\|_2^2 - \langle\psi,\rho\rangle. \tag{2.99}
$$

Then, it follows that

$$
\begin{aligned}
L|_{\rho_\alpha = \frac{e^{-\frac{\beta}{2}\alpha\psi}}{\int_\Omega e^{-\frac{\beta}{2}\alpha\psi}}} &= J \\
L|_{\psi=(-\Delta_D)^{-1}\rho} &= -\frac{2}{\beta}I.
\end{aligned}
$$

Putting $v = -\frac{\beta}{2}\psi$ and $\lambda = -\frac{\beta}{2}$ in (2.95), we obtain

$$
-\Delta v = \lambda\int_{[-1,1]}\frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}}P(d\alpha) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{2.100}
$$

with the variational functional

$$
J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\int_{[-1,1]}\log\left(\int_\Omega e^{\alpha v}\right)P(d\alpha) \tag{2.101}
$$

defined for $v \in H_0^1(\Omega)$. The Moser-Onofri-Hong inequality[147,206,240] implies

$$
\log\left(\int_\Omega e^{\alpha v}\right) \le \frac{1}{16\pi}\|\nabla v\|_2^2 + K
$$

for $\alpha \in [-1,1]$, $v \in H_0^1(\Omega)$, where K is a constant determined by Ω. The inequality

$$
\inf_{v\in H_0^1(\Omega)} J_{8\pi}(v) > -\infty
$$

derived from the Moser-Onofri-Hong inequality is improved as follows and then we obtain the existence of the solution to (2.95) for $\beta > -2\lambda_*$.

Theorem 2.4. *For $J_\lambda = J_\lambda(v)$ defined by (2.101), it holds that*

$$\inf_{v\in H_0^1(\Omega)} J_{\lambda_*}(v) > -\infty \tag{2.102}$$

for

$$\lambda_* = \frac{8\pi}{\int_{[-1,1]} \alpha^2 dP(\alpha)}. \tag{2.103}$$

Proof: From the Pohozaev-Trudinger-Moser inequaliy,[206,257,325] it holds that

$$\int_\Omega e^{4\pi w^2} \leq C \qquad \text{for} \qquad \|\nabla w\|_2 \leq 1. \tag{2.104}$$

Using this property with

$$\alpha v \leq 4\pi \frac{(\alpha v)^2}{\|\nabla(\alpha v)\|_2^2} + \alpha^2 \frac{\|\nabla v\|_2^2}{16\pi}$$

for $v \in H_0^1(\Omega) \setminus \{0\}$, we obtain

$$\int_{[-1,1]} \log\left(\int_\Omega e^{\alpha v}\right) P(d\alpha) \leq \frac{\|\nabla v\|_2^2}{16\pi} \int_{[-1,1]} \alpha^2 P(d\alpha) + K$$

with a constant K determined by Ω which implies that

$$J_\lambda(v) \geq \left(\frac{1}{2} - \frac{\lambda}{16\pi}\int_{[-1,1]} \alpha^2 P(\alpha)\right) \|\nabla v\|_2^2 - \lambda K$$

and hence (2.102). □

In the *neutral* case defined by

$$P(d\alpha) = \frac{1}{2}\{\delta_{-1}(d\alpha) + \delta_1(d\alpha)\}, \tag{2.105}$$

it follows that

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda}{2}\left[\log\left(\int_\Omega e^v\right) + \log\left(\int_\Omega e^{-v}\right)\right].$$

We have $\lambda_* = 8\pi$ for λ_* defined by (2.103), but actually it holds that

$$\inf_{v\in H_0^1(\Omega)} J_{16\pi}(v) > -\infty,$$

see.[236,285] The mean field equation (2.100) to (2.105), on the other hand, is formulated by

$$-\Delta v = \frac{\lambda}{2}\left(\frac{e^v}{\int_\Omega e^v} - \frac{e^{-v}}{\int_\Omega e^{-v}}\right) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{2.106}$$

This equation was derived by the above described method[159] and also by the other method of direct calculation[258] using the micro-canonical ensemble. From the latter method, we obtain (2.95) with $\frac{\beta}{2}$ replaced by β, see.[269] There will be a sharper form of inequality (2.102) regarding the dual inequality, see.[62,232,284,285]

Stochastic Intensity

It is shown by the method of minimal free energy using the canonical ensemble[221] that if the intensities of the vortices are independent random variables $\alpha \in [-1,1]$ subject to the the same distribution $P(d\alpha)$, then the mean field equation is defined by

$$\rho = \frac{\int_{[-1,1]} \alpha e^{-\alpha\beta\psi} P(d\alpha)}{\int_{[-1,1]} \left(\int_\Omega e^{-\alpha\beta\psi}\right) P(d\alpha)}$$
$$\psi = (-\Delta_D)^{-1}\rho. \tag{2.107}$$

Thus in the neutral case of (2.105), slightly different from (2.106), it holds that

$$-\Delta v = \lambda \left(\frac{e^v - e^{-v}}{\int_\Omega e^v + \int_\Omega e^{-v}} \right) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

and the associated variational functional is

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v + \int_\Omega e^{-v}\right)$$

defined for $v \in H_0^1(\Omega)$. This case is also obtained by the method of maximal entropy.[84] In fact, in this case we put

$$\tilde{x}_i = (x_i, \alpha^i) \in \Omega \times [-1,1] = \tilde{\Omega}$$

because the intensity $\alpha_i = \alpha^i\alpha$ is also a random variable besides the position x_i at each vortex point. The Hamiltonian is described by

$$H^n(\tilde{X}^n) = \frac{1}{2}\alpha^2 \sum_{i=1}^n \sum_{j=1, j\neq i}^n \alpha^i\alpha^j G(x_i,x_j) + \frac{1}{2}\alpha^2 \sum_{i=1}^n (\alpha^i)^2 R(x_i),$$

and, therefore, dividing

$$\tilde{X}^n = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{\Omega}^n$$

into

$$\tilde{X}^k = (\tilde{x}_1, \ldots, \tilde{x}_i) \in \tilde{\Omega}^k$$
$$\tilde{X}_{n-k} = (\tilde{x}_{k+1}, \ldots, \tilde{x}_n) \in \tilde{\Omega}^{n-k},$$

we obtain

$$H^n(\tilde{X}^n) = H^k(\tilde{X}^k) + H^{n-k}(\tilde{X}_{n-k}) + \sum_{i=1}^k \sum_{j=k+1}^n \alpha^i\alpha\alpha^j\alpha G(x_i,x_j).$$

The canonical measure, on the other hand, is described by

$$\mu^n(d\tilde{X}^n) = \frac{e^{-\tilde{\beta}H^n(\tilde{X}^n)}d\tilde{X}^n}{Z(n,\tilde{\beta})}$$
$$Z(n,\beta) = \int_{\tilde{\Omega}^n} e^{-\tilde{\beta}H^n(\tilde{X}^n)}d\tilde{X}^n,$$

and, therefore, the k-point reduced pdf is defined by

$$\rho_k^n = \int_{\tilde{\Omega}^{n-k}} \mu^n(d\tilde{X}^{n-k}) = \frac{e^{-\tilde{\beta}H^k(\tilde{X}^k)}}{Z(n)} \cdot$$
$$\cdot \int_{\tilde{\Omega}^{n-k}} e^{-\tilde{\beta}H^{n-k}(\tilde{X}_{n-k})}e^{-\tilde{\beta}\alpha^2\sum_{i=1}^{k}\sum_{j=k+1}^{n}\alpha^i\alpha^j G(x_i,x_j)}d\tilde{X}_{n-k}.$$

Using

$$d\tilde{X}_{n-k} = Z(n-k)e^{\frac{\tilde{\beta}n}{n-k}H^{n-k}(\tilde{X}_{n-k})}\mu^{n-k}(d\tilde{X}_{n-k}),$$

we deduce

$$\rho_k^n = \frac{Z(n-k)}{Z(n)}e^{-\tilde{\beta}H^k(\tilde{X}^k)} \cdot$$
$$\cdot \int_{\tilde{\Omega}^{n-k}} e^{\frac{\tilde{\beta}k}{n-k}H^{n-k}(\tilde{X}_{n-k})}e^{-\tilde{\beta}\alpha^2\sum_{i=1}^{k}\sum_{j=k+1}^{n}G(x_i,x_j)}\mu^{n-k}(d\tilde{X}_{n-k}),$$

and, in particular,

$$\rho_1^n(\tilde{x}_1) = \frac{Z(n-1)}{Z(n)}\exp\left[-\frac{\beta(\alpha^1)^2}{2n}R(x_1)\right]$$
$$\cdot\int_{\tilde{\Omega}^{n-k}}\exp\left[\frac{\beta}{2n(n-1)}\sum_{i=2}^{n}\sum_{j=2,j\neq i}^{n}\alpha^i\alpha^j G(x_i,x_j)\right.$$
$$\left.-\frac{\beta}{2n(n-1)}\sum_{i=2}^{n}(\alpha^i)^2R(x_i)\right]\cdot\exp\left[-\frac{\beta}{n}\sum_{j=2}^{n}\alpha^j G(x_i,k_j)\right]\mu^{n-1}(d\tilde{X}^{n-1}).$$

Thus assuming $\rho_1^n \rightharpoonup \rho_1$ and the propagation of chaos, we obtain

$$\rho_1(\tilde{x}_1) = Z^{-1}\exp\left[\frac{1}{2}\iint_{\tilde{\Omega}\times\tilde{\Omega}}\alpha^1\alpha^2G(x_1,x_2)\rho_1(\tilde{x}_1)\rho_1(\tilde{x}_2)d\tilde{x}_1d\tilde{x}_2\right]$$
$$\cdot\ \exp\left[-\beta\alpha^1\int_{\tilde{\Omega}}\alpha^2G(x_1,x_2)\rho_1(\tilde{x}_2)d\tilde{x}_2\right]$$

for $Z = \lim_{n\to\infty}\frac{Z(n)}{Z(n-1)}$, which guarantees

$$\rho_1(\tilde{x}_1) = \frac{\exp\left[-\beta\alpha^1\int_{\tilde{\Omega}}\alpha^2G(x_1,x_2)\rho_1(\tilde{x}_2)d\tilde{x}_2\right]}{\int_{\tilde{\Omega}}\exp\left[-\beta\alpha^1\int_{\tilde{\Omega}}\alpha^2G(x_1,x_2)\rho_1(\tilde{x}_2)\right]d\tilde{x}_1}.$$

This equality means

$$\rho_1(\tilde{x}_1) = \frac{\exp\left[\beta\alpha^1\psi(x)\right]}{\int_{\tilde{\Omega}}\exp\left[\beta\alpha^1\psi(x)\right]d\tilde{x}_1}$$

for the mean field stream function

$$\psi(x_1) = \int_{\tilde{\Omega}} \alpha^2 G(\cdot, x_2) \rho_1(\tilde{x}_2) d\tilde{x}_2, \tag{2.108}$$

and hence

$$-\Delta\psi = \int_{[-1,1]} \alpha^1 \rho_1(\tilde{x}_1) P(d\alpha^1) \tag{2.109}$$

because α^i $(i = 1, 2, \dots)$ has the same distribution functions. Then, (2.107) holds by (2.108) and (2.109).
The mean field equation (2.107) implies

$$\begin{aligned} &\log\rho_\alpha + \beta(-\Delta_D)^{-1}\rho = \text{constant} \\ &\rho_\alpha = \frac{e^{-\alpha\beta\psi}}{\int_{[-1,1]} \left(\int_\Omega e^{-\alpha\beta\psi}\right) P(d\alpha)}, \end{aligned} \tag{2.110}$$

and then it holds that

$$\begin{aligned} &\rho_\alpha \geq 0, \ \int_{[-1,1]} \rho_\alpha P(d\alpha) = 1 \\ &\rho = \int_{[-1,1]} \alpha\rho_\alpha P(d\alpha). \end{aligned} \tag{2.111}$$

Then, (2.110) is the Euler-Lagrange equation for

$$I(\oplus\rho_\alpha) = \int_\Omega \left(\int_{[-1,1]} \rho_\alpha(\log\rho_\alpha - 1) P(d\alpha)\right) + \frac{\beta}{2}\left\langle(-\Delta_D)^{-1}\rho, \rho\right\rangle$$

constrained by (2.111).
The relation (2.107) also implies

$$-\Delta\psi = \frac{\int_{[-1,1]} \alpha e^{-\alpha\beta\psi} P(d\alpha)}{\int_{[-1,1]} \left(\int_\Omega e^{-\alpha\beta\psi}\right) P(d\alpha)} \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega \tag{2.112}$$

which is provided with the variational function

$$J(\psi) = \frac{1}{2}\|\nabla\psi\|_2^2 + \frac{1}{\beta}\log\left(\int_\Omega \left[\int_{[-1,1]} e^{-\alpha\beta\psi} P(d\alpha)\right]\right) + \frac{1}{\beta}$$

defined for $\psi \in H_0^1(\Omega)$. Then, introducing the Lagrangian

$$L(\oplus\rho_\alpha, \psi) = -\frac{1}{\beta}\int_\Omega \left[\int_{[-1,1]} \rho_\alpha(\log\rho_\alpha - 1) P(d\alpha)\right] + \frac{1}{2}\|\nabla\psi\|_2^2 - \langle\psi, \rho\rangle,$$

we obtain the unfolding

$$\begin{aligned} &L|_{\rho_\alpha \frac{e^{-\alpha\beta\psi}}{\int_{[-1,1]}\left(\int_\Omega e^{-\alpha\beta\psi}\right)P(d\alpha)}} = J \\ &L|_{\psi = (-\Delta_D)^{-1}\rho} = -\frac{1}{\beta} I. \end{aligned}$$

Using $v = \beta\psi$ and $\lambda = -\beta$, we can write (2.112) as

$$-\Delta v = \frac{\int_{[-1,1]} \alpha e^{\alpha v} P(d\alpha)}{\int_{[-1,1]} \left(\int_\Omega e^{\alpha v}\right) P(d\alpha)} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2.113}$$

with the variational function

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega \left[\int_{[-1,1]} e^{\alpha v} P(d\alpha)\right]\right) \tag{2.114}$$

defined for $v \in H_0^1(\Omega)$. The following theorem, obtained similarly to (2.102), guarantees the existence of the solution to (2.113) for $\lambda < 8\pi$. Similar to the deterministic case (2.98), this value $\lambda = 8\pi$ may not be optimal for $J_\lambda(v)$, $v \in H_0^1(\Omega)$ defined by (2.114) to be bounded from below.

Theorem 2.5 ([221]). *It holds that*

$$\inf_{v \in H_0^1(\Omega)} J_{8\pi}(v) > -\infty \tag{2.115}$$

for $J_\lambda = J_\lambda(v)$ defined by (2.114).

Proof: Given $v \in H_0^1(\Omega) \setminus \{0\}$, we apply the Pohozaev-Trudinger-Moser inequality (2.104) to

$$\alpha v \leq \frac{1}{16\pi}\|\nabla v\|_2^2 + 4\pi\alpha^2 \cdot \frac{v^2}{\|\nabla v\|_2^2},$$

where $\alpha \in [-1,1]$ and $C > 0$ is a constant determined by Ω. This inequality implies

$$\log\left(\int_\Omega \left[\int_{[-1,1]} e^{\alpha v} P(d\alpha)\right]\right) \leq \frac{1}{16\pi}\|\nabla v\|_2^2 + K$$

for $v \in H_0^1(\Omega)$, with the constant K determined by Ω. Then (2.115) follows. □

2.2.5. Kramers-Moyal Expansion

The last two paragraphs of this section are concerned with the macroscopic description of the non-stationary mean field equation based on the microscopic and mezoscopic overviews, respectively.

The first equation of (1.8) is called the *Smoluchowski* equation. First, *master equation* describes the particle transport using the conditional probability. Then, the *Kramers equation* is obtained by the *Kramers-Moyal expansion*. The adiabatic limit of this equation is called the Smoluchowski equation, whereby the conditional probability limit casts the particle distribution. Such a kinetic theory that derives from the drift-diffusion equation is adapted by several authors in the context of chemotaxis in biology,[5,144,246,248,290,293] while the other argument applies the method of renormalization directly to the master equation.[247] This paragraph describes the former classical approach. The space is assumed to be one-dimensional, to make the description simple. See[247] for the latter approach.

Master Equation

Let $P(x_2,t_2 \mid x_1,t_1)$ be the conditional existence probability of a particle at the position $x = x_2$ and the time $t = t_2$ that was at $x = x_1$ for $t = t_1$. Then it holds that

$$P(x_2,t_1+\Delta t \mid x_1,t_1) = F\delta(x_2 - x_1) + \Delta t \cdot W(x_1 \to x_2) \tag{2.116}$$

for $0 < \Delta t \ll 1$, where the first and the second terms of the right-hand side describe the state variation and the transient probability, respectively. Using

$$\int dx_2 P(x_2,t_2 \mid x_1,t_1) = 1,$$

we obtain

$$F + \Delta t \int dx' W(x_1 \to x') = 1$$

and, therefore,

$$\begin{aligned} P(x_2,t_1+\Delta t \mid x_1,t_1) = \delta(x_2 - x_1)\left\{1 - \Delta t \int dx' W(x_1 - x')\right\} \\ +\Delta t \cdot W(x_1 \to x_2). \end{aligned} \tag{2.117}$$

Now, we apply *Chapman-Kolmogorov's equation*

$$P(x_3,t_3 \mid x_1,t_1) = \int P(x_3,t_3 \mid x_2,t_2) P(x_2,t_2 \mid x_1,t_1) dx_2\, dt_2$$

to (2.117), and then it follows that

$$\begin{aligned} &P(x_3,t_2+\Delta t \mid x_1,t_1) \\ &= \int dx_2 \left[\delta(x_3 - x_2)\left\{1 - \Delta t \int dx' W(x_2 \to x')\right\} + \Delta t \cdot W(x_2 \to x_3)\right] \\ &\quad \cdot P(x_2,t_2 \mid x_1,t_1) \\ &= \left\{1 - \Delta t \int dx' W(x_3 - x')\right\} P(x_3,t_2 \mid x_1,t_1) \\ &\quad + \int dx_2 W(x_2 \to x_3) P(x_2,t_2 \mid x_1,t_1) \Delta t. \end{aligned}$$

This equality implies

$$\begin{aligned} &\frac{1}{\Delta t}\{P(x_3,t_2+\Delta t \mid x_1,t_1) - P(x_3,t_2 \mid x_1,t_1)\} \\ &= -\int dx' W(x_3 \to x') P(x_3,t_2 \mid x_1,t_1) \\ &\quad + \int dx_2 W(x_2 \to x_3) P(x_2,t_2 \mid x_1,t_1), \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{\partial}{\partial t} P(x,t \mid x_1,t_1) = -\int dx' W(x \to x') P(x,t \mid x_1,t_1) \\ + \int dx' W(x' \to x) P(x',t \mid x_1,t_1). \end{aligned} \tag{2.118}$$

This formula is called the master equation, and describes the time variation of the existence probability P controlled by the transient probability W, see.[282]

Kramers Equation

The right-hand side of (2.118) is described by

$$-\int dyW(x\to x+y)P(x,t\mid x_1,t_1)+\int dyW(x-y\to x)P(x-y,t\mid x_1,t_1),$$

and we apply the formal Taylor's expansion,

$$f(x+a)=\sum_{k=0}^{\infty}\frac{a^k}{k!}\partial^k f(x)=[\exp(a\partial_x)]\,f(x),$$

to the second term. Thus it holds that

$$\begin{aligned}&\int dyW(x-y\to x)P(x-y,t\mid x_1,t_1)\\&\qquad=\int dy\,[\exp(-y\partial_x)]\,W(x\to x+y)P(x,t\mid x_1,t_1),\end{aligned}$$

and hence

$$\begin{aligned}&\frac{\partial}{\partial t}P(x,t\mid x_1,t_1)\\&=-\int dy\,[1-\exp(-y\partial_x)]\,W(x\to x+y)P(x,t\mid x_1,t_1)\\&=\int dy\sum_{k=1}^{\infty}\frac{1}{k!}\,(-y\partial_x)^k\,W(x\to x+y)P(x,t\mid x_1,t_1)\\&=\sum_{k=1}^{\infty}\frac{1}{k!}\,(-\partial_x)^k\,C_k(x)P(x,t\mid x_1,t_1)\end{aligned}\tag{2.119}$$

for

$$C_k(x)=\int W(x\to x+y)y^k dy.$$

Equality (2.119) is called the Kramers-Moyal expansion. Here, we apply (2.117) and obtain

$$\begin{aligned}&\int P(x+y,t+\Delta t\mid x,t)y^k dy\\&\quad=\int\left[\delta(y)\left\{1-\Delta t\int dx'W(x\to x')\right\}+\Delta tW(x\to x+y)\right]y^k dy\\&\quad=\Delta t\int W(x\to x+y)y^k dy\end{aligned}$$

because $k\geq 1$. Thus it holds that

$$\begin{aligned}C_k(x)&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int P(x+y,t+\Delta t\mid x,t)y^k dy\\&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int[P(x+y,t+\Delta t\mid x,t)-P(x+y,t\mid x,t)]\,y^k dy,\end{aligned}$$

and henceforth,

$$\int [P(x+y,t+\Delta t \mid x,t) - P(x+y,t \mid x,t)] y^k dy$$

is denoted by

$$\left\langle [x(t+\Delta t) - x(t)]^k \right\rangle_{x(t)=x}$$

because it indicates the mean value of the k-th moment constrained by $x(t) = x$. Thus we obtain

$$C_k(x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle [x(t+\Delta t) - x(t)]^k \right\rangle_{x(t)=x}.$$

Fluctuation-Friction

The *Langevin equation* describes the motion of a particle subject to the fluctuation-friction and is formulated by

$$\begin{aligned} \frac{dx}{dt} &= v \\ m\frac{dv}{dt} &= -m\gamma v + R(t) + mF(x) \end{aligned}$$

see,[80] where m, γ, $R(t)$, and $F(x)$ denote the mass, the friction coefficient, the fluctuation, and the outer force, respectively. Then we can derive the Kramers-Moyal expansion of the master equation concering the conditional probability $P(x,v,t \mid x_1,v_1,t_1)$.
In fact, replacing this relation by

$$\begin{aligned} v(t) &= v(s)e^{-\gamma(t-s)} + \int_s^t e^{-\gamma(t-\tau)} \left[\frac{R(\tau)}{m} + F(x(\tau)) \right] d\tau \\ x(t) &= x(s) + \int_s^t v(\tau) d\tau \end{aligned}$$

for $t \geq s$, we obtain

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle x(t+\Delta t) - x(t) \rangle_{v(t)=v,\ x(t)=x} &= v \\ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle [x(t+\Delta t) - x(t)]^2 \right\rangle_{v(t)=v,\ x(t)=x} &= 0 \end{aligned} \tag{2.120}$$

from the second equation. Next, it holds that

$$v(t+\Delta t) \approx (1-\gamma\Delta t)v(t) + \int_t^{t+\Delta t} e^{-\gamma(t+\Delta t-\tau)} \frac{R(\tau)}{m} d\tau + F(x(t))\Delta t,$$

or

$$\begin{aligned} v(t+\Delta t) - v(t) \approx{}& (-\gamma v(t) + F(x(t)))\,\Delta t \\ &+ \int_t^{t+\Delta t} e^{-\gamma(t+\Delta t-\tau)} \frac{R(\tau)}{m} d\tau \end{aligned} \tag{2.121}$$

by the first equation.

Since $\langle R(\tau)\rangle = 0$, it holds that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle v(t+\Delta t)-v(t)\rangle_{v(t)=v,\ x(t)=x} = -\gamma v + F(x). \tag{2.122}$$

Similarly, we obtain

$$[v(t+\Delta t)-v(t)]^2 \approx 2(\gamma v(t)+F(x(t)))\Delta t\cdot\int_t^{t+\Delta t} e^{-\gamma(t+\Delta t-\tau)}\frac{R(\tau)}{m}d\tau$$
$$+\left|\int_t^{t+\Delta t} e^{-\gamma(t+\Delta t-\tau)}\frac{R(\tau)}{m}d\tau\right|^2$$

and hence

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle [v(t+\Delta t)-v(t)]^2\right\rangle_{v(t)=v,\ x(t)=x}$$
$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot\frac{1}{m^2}\int_t^{t+\Delta t}\int_t^{t+\Delta t} e^{-\gamma(2t+2\Delta t-t_1-t_2)}\langle R(t_1)R(t_2)\rangle\, dt_1 dt_2.$$

If the fluctuation is a *white noise*, then its correlation is defined by

$$\langle R(t_1)R(t_2)\rangle = 2D\delta(t_1-t_2)$$

with $D>0$, and then it follows that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot\frac{1}{m^2}\int_t^{t+\Delta t}\int_t^{t+\Delta t} e^{-\gamma(2t+2\Delta t-t_1-t_2)}\langle R(t_1)R(t_2)\rangle\, dt_1 dt_2 = \frac{2D}{m^2}.$$

We obtain

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle |v(t+\Delta t)-v(t)|^2\rangle_{v(t)=v, x(t)=x} = \frac{2D}{m^2}, \tag{2.123}$$

and, therefore,

$$C_1(x,v) = (v, -\gamma v + F(x))$$
$$C_2(x,v) = \left(0, 2D/m^2\right)$$

by (2.120), (2.122), and (2.123). Hence (2.119) is reduced to the *Kramers equation*,

$$\frac{\partial}{\partial t}P(x,v,t\mid x_1,v_1,t_1) = \left[-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}(-F(x)+\gamma v) + \frac{D}{m^2}\frac{\partial^2}{\partial v^2}\right]$$
$$\cdot P(x,v,t\mid x_1,v_1,t) \tag{2.124}$$

by neglecting the higher terms of $k\geq 3$. This equation is called the *Fokker-Planck equation* if it is uniform in x:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[-F+\gamma v+\frac{D}{m^2}\frac{\partial}{\partial v}\right]P. \tag{2.125}$$

Smoluchowski Equation

If K and T denotes the force field and the temperature, then we obtain $F = \frac{K}{m}$ and $D = mkT$. In this case, the Kramers equation (2.124) is described by

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{K}{m}\frac{\partial P}{\partial v} - v\frac{\partial P}{\partial x} + \gamma\frac{\partial}{\partial v}\left(vP + \frac{kT}{m}\frac{\partial P}{\partial v}\right) \\ &= \gamma\frac{\partial}{\partial v}\left(vP + \frac{kT}{m}\frac{\partial P}{\partial v} - \frac{K}{\gamma m}P + \frac{kT}{\gamma m}\frac{\partial P}{\partial x}\right) - \frac{kT}{m}\frac{\partial^2 P}{\partial v\partial x} - v\frac{\partial P}{\partial x} \\ &= \gamma\left(\frac{\partial}{\partial v} - \frac{1}{\gamma}\frac{\partial}{\partial x}\right)\left(vP + \frac{kT}{m}\frac{\partial P}{\partial v} - \frac{K}{\gamma m}P + \frac{kT}{\gamma m}\frac{\partial P}{\partial x}\right) \\ &\quad -\frac{\partial}{\partial x}\left(\frac{K}{\gamma m}P - \frac{kT}{\gamma m}\frac{\partial P}{\partial x}\right).\end{aligned}$$

Using

$$\begin{aligned} q &= x + \frac{v}{\gamma} \\ p &= v, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial v} &= \frac{1}{\gamma}\frac{\partial}{\partial q} - \frac{\partial}{\partial p} \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial q}, \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{\partial P}{\partial t} &= -\gamma\frac{\partial A}{\partial p} - \frac{\partial}{\partial q}\left(\frac{K}{\gamma m}P - \frac{kT}{\gamma m}\frac{\partial P}{\partial q}\right) \\ A &= vP + \frac{kT}{m}\frac{\partial P}{\partial v} - \frac{K}{\gamma m}P + \frac{kT}{\gamma m}\frac{\partial P}{\partial x}. \end{aligned}$$

Thus if

$$\begin{aligned} &\int P\left(q - \frac{p}{\gamma}, p, t\right) dp \sim f(x,t)|_{x=q-\frac{p}{\gamma}} \\ &\int K\left(q - \frac{p}{\gamma}\right) P\left(q - \frac{p}{\gamma}, p, t\right) dp \sim K(x)f(x,t)|_{x=q-\frac{p}{\gamma}}, \end{aligned} \tag{2.126}$$

then the Smoluchowski equation

$$\frac{\partial f}{\partial t} = \frac{1}{\gamma m}\frac{\partial}{\partial x}\left(-Kf + kT\frac{\partial f}{\partial x}\right)$$

can take place of (2.124), where

$$f(x,t) = \int P(x,v,t)dv.$$

From the Kramers equation, the existence probability $P(x,v,t)$ is almost supported in $|v| = O\left(\left(\frac{\gamma kT}{m}\right)^{1/2}\right)$, and, therefore, the approximation (2.126) is valid if the variaions of $P = P(x,v,t)$ and $K = K(x)$ in x are uniformly small for $|\Delta x| = O\left(\left(\frac{kT}{\gamma m}\right)^{1/2}\right)$.

2.2.6. Kinetic Theory

The Smoluchowski-Poisson equation (1.8) arises also in the kinetic theory in accordance with the Boltzmann entropy.[51] A relative is the degenerate parabolic equation of which stationary state is provided with the quantized blowup mechanism.[308]
First, the parabolic-elliptic system

$$\begin{aligned} &\mu_t = \nabla[D_* \cdot (\nabla p + \mu \nabla \varphi)] \\ &\Delta \varphi = \mu \qquad \text{in } \Omega \times (0,T) \end{aligned} \tag{2.127}$$

is derived as the hydrodynamical limit of self-gravitating particles. Here, $\mu = \mu(x,t) \geq 0$ is the function describing particle density at $(x,t) \in \Omega \times (0,T)$, $\Omega \subset \mathbf{R}^n$, $n \geq 2$, a domain, $\varphi = \varphi(x,t)$ the gravitational potential generated by μ, and $p \geq 0$ the pressure determined by the density-pressure relation

$$p = p(\mu, \theta). \tag{2.128}$$

If Ω has the boundary $\partial\Omega$, the null-flux boundary condition

$$(\nabla p + \mu \nabla \varphi) \cdot \nu = 0$$

is imposed with ν denoting the outer unit normal vector so that the total mass

$$\lambda = \int_\Omega \mu(x,t)dx$$

is conserved during the evolution.
Details of the derivation are as follows. First, the density of particles at $(x,t) \in \Omega \times (0,T)$ moving at the velocity v is denoted by $0 \leq f = f(x,v,t)$. It is subject to the kinetic equation

$$f_t + v \cdot \nabla_x f - \nabla \varphi \cdot \nabla_v f = -\nabla_v \cdot j$$

with the general dissipation flux term $-\nabla_v \cdot j$. This flux term is determined by the maximum entropy production principle, so that f maximize the local entropy $S = \int_{\mathbf{R}^n} s(f(x,v,t))dv$ under the constraint

$$\begin{aligned} \mu(x,t) &= \int_{\mathbf{R}^n} f(x,v,t)dv \\ p(x,t) &= \frac{1}{n} \int_{\mathbf{R}^n} |v|^2 f(x,v,t)dv. \end{aligned}$$

Averaging f over the velocities $v \in \mathbf{R}^n$, and then the passage to the limit of large friction or large times lead to the first equation of (2.127) in the (x,t) space, see.[51] We have, thus, several mean field equations according to the entropy function $s(f)$ subject to the law of partition of macroscopic states of particles into mezoscopic states, that is the entropies of Boltzmann, Fermi-Dirac, Bose-Einstein, and so forth.
System (2.127) with (2.128) is still under-determined, and there are several theories to prescribe the temperature θ. In the canonical statistics one takes the iso-thermal setting, and hence the temperature $\theta > 0$ is a constant. In the micro-canonical statistics, on the other hand, $\theta = \theta(t) > 0$ is the function of t, where

$$E = \frac{n}{2} \int_\Omega p dx + \frac{1}{2} \int_\Omega \mu \varphi dx$$

is the prescribed total energy independent of t.
If Rényi-Tsallis' entropy

$$S = \frac{-1}{q-1}\int_{\mathbf{R}^n}(f^q - f)dv$$

is adopted, then (2.128) becomes

$$p = \kappa\theta^{1-\frac{\gamma n}{2}}\mu^{1+\gamma},$$

where $\kappa > 0$ is a constant and $\frac{1}{\gamma} = \frac{1}{q-1} + \frac{n}{2}$, see[26,52] for details. By normalizing constants and assuming $\Omega = \mathbf{R}^n$, then we can reduce (2.127) to the degenerate parabolic equation

$$\begin{aligned} u_t &= \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u) \\ u &\geq 0 \qquad\qquad \text{in } \mathbf{R}^n \times (0,T) \end{aligned} \tag{2.129}$$

in the iso-thermal setting, where the new unknown u is a positive constant times μ, $\frac{1}{m-1} = \frac{1}{q-1} + \frac{n}{2}$, and

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}}$$

with ω_{n-1} denoting the $(n-1)$ dimensional volume of the boundary of the unit ball in $\mathbf{R}^n$ if $n \geq 3$ and

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|} \tag{2.130}$$

if $n = 2$.
When $n = 3$ and $q = \frac{5}{3}$, the case $m = 2 - \frac{2}{n} = \frac{4}{3}$ actually arises to (2.129). As we shall show by the scaling argument, equation (2.129) of this exponent m is, mathematically, a higher-dimensional version of the Smoluchowski-Poisson equation associated with the Boltzmann entropy in two-space dimensions. This two-dimensional equation is given by

$$\begin{aligned} u_t &= \Delta u - \nabla\cdot(u\nabla\Gamma * u) \\ u &\geq 0 \qquad\qquad \text{in } \mathbf{R}^2 \times (0,T) \end{aligned} \tag{2.131}$$

with $\Gamma = \Gamma(x)$ defined by (2.130). It is thus a relative to (1.8), the simplified system of chemotaxis, associated with the total mass conservation $\|u(t)\|_1 = \|u_0\|_1$ and the decrease of the free energy,

$$\mathscr{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u dxdx',$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary, ν the outer unit normal vector, $u\otimes u = u(x,t)u(x',t)$, and $G = G(x,x')$ the Green's function to (1.15).
Similarly to (1.8), there is a collapse formation with the quantized mass of the blowup solution in finite time to (2.129), provided that

$$u_0 = u|_{t=0} \in X = L^1(\mathbf{R}^2,(1+|x|^2)dx)\cap L^\infty(\mathbf{R}^2)\cap H^1(\mathbf{R}^2).$$

In fact, (2.131) is well-posed in this function space X locally in time, and it follows that

$$\frac{dI}{dt} = 4\lambda - \frac{\lambda^2}{2\pi}$$

for

$$I = \int_{\mathbf{R}^2} |x|^2 u$$
$$\lambda = \int_{\mathbf{R}^2} u.$$

We have $T = T_{\max} < +\infty$ and $T = +\infty$ if $\lambda > 8\pi$ and $\lambda < 8\pi$, respectively. In the case of $T = T_{\max} < +\infty$, it holds that

$$\limsup_{t\uparrow T} I(t) < +\infty$$

and this property guarantees the boundedness of the blowup set in $\mathbf{R}^2$ from the ε-regularity and a Chebyshev type inequality. Then, we obtain an analogous result of (1.11)-(1.13), similarly, so that if $T < +\infty$ then

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathscr{S}} 8\pi\delta_{x_0}(dx) + f(x)dx$$

in $\mathscr{M}(\mathbf{R}^2 \cup \{\infty\})$ as $t \uparrow T$, where $\mathscr{S}$ is the blowup set of u, which is finite, $0 \le f = f(x) \in L^1(\mathbf{R}^2) \cap C(\mathbf{R}^2 \setminus \mathscr{S})$, and $\mathbf{R}^2 \cup \{\infty\}$ is the one-point compactification of $\mathbf{R}^2$.
Equation (2.129) is also a model (B) equation associated with the *free energy*

$$\mathscr{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} - \frac{1}{2}\langle \Gamma * u, u\rangle. \tag{2.132}$$

It is actually the free energy formulating the equilibrium of the Euler-Poisson equation, see §2.1.2. In fact, we have

$$\delta\mathscr{F}(u)[v] = \frac{d}{ds}\mathscr{F}(u+sv)\bigg|_{s=0} = \left\langle v, u^{m-1} - \Gamma * u\right\rangle,$$

where $\langle\ ,\ \rangle$ denotes the L^2-inner product. Identifying $\mathscr{F}(u)$ with $u^{m-1} - \Gamma * u$, we can write (2.129) as

$$\begin{aligned} u_t &= \nabla\cdot\left(\frac{m-1}{m}\nabla u^m - u\nabla\Gamma * u\right) \\ &= \nabla\cdot u\nabla\delta\mathscr{F}(u) \qquad \text{in } \mathbf{R}^n \times (0,T). \end{aligned} \tag{2.133}$$

From this form of (2.133), it is easy to infer, at least formally, the total mass conservation

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda \tag{2.134}$$

and the decrease of the free energy

$$\begin{aligned} \frac{d}{dt}\mathscr{F}(u) &= -\int_{\mathbf{R}^n} u\,|\nabla\delta\mathscr{F}(u)|^2 \\ &= -\int_{\mathbf{R}^n} u\left|\nabla(u^{m-1} - \Gamma * u)\right|^2 \\ &\le 0. \end{aligned} \tag{2.135}$$

See[298,300,311] for the rigorous proof and the notion of the weak solution.
Regarding (2.134)-(2.135), we formulate the stationary state by

$$u^{m-1} - \Gamma * u = \text{constant in } \{u > 0\}$$
$$\int_{\mathbf{R}^n} u = \lambda. \tag{2.136}$$

If the above constant is denoted by c, then $v = \Gamma * u + c$ satisfies

$$-\Delta v = v_+^q \qquad \text{in } \mathbf{R}^n$$
$$\int_{\mathbf{R}^n} v_+^q = \lambda, \tag{2.137}$$

where $m = 1 + \frac{1}{q}$. This constant c may depend on the connected component of $\{u > 0\}$ at this moment. We obtain, thus, a relative to (2.26) for the bounded domain case, see also §2.4.3.
Problem (2.137) is invariant under the scaling transformation

$$v(x) \mapsto v_\mu(x) = \mu^\gamma v(\mu x) \tag{2.138}$$

if and only if $\gamma = n-2$ and $q = \frac{1}{m-1} = \frac{n}{n-2}$, that is $m = 2 - \frac{2}{n}$, where $\mu > 0$ is a constant. If this exponent is the case, conversely, problem (2.138) admits a family of solutions, each of which is necessarily radially symmetric and has compact support, see.[331] Then, we recall the three dimensional case described in §2.1.2, and define the normalized solution $v_* = v_*(x)$ to (2.137) and the quantized mass $\lambda_* > 0$ by

$$-\Delta v_* = v_{*+}^q, \ v_* \le v_*(0) = 1 \quad \text{in } \mathbf{R}^n$$
$$\lambda_* = \int_{\mathbf{R}^n} v_{*+}^q. \tag{2.139}$$

Then, we obtain the threshold for the existence of the solution global in time, see also.[299,300]

Theorem 2.6 ([311]). *If $u_0 = u_0(x)$ is the initial value satisfying*

$$0 \le u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$$
$$u_0^m \in H^1(\mathbf{R}^n)$$
$$\int_{\mathbf{R}^n} |x|^2 u_0 < +\infty \tag{2.140}$$

and $\|u_0\|_1 < \lambda_*$, then $T = +\infty$ holds in (2.129) for $m = 2 - \frac{2}{n}$. Each $\lambda > \lambda_*$, on the other hand, takes $u_0 = u_0(x)$ such that (2.140), $\|u_0\|_1 = \lambda$, and $T < +\infty$.

The non-stationary problem (2.129) with $m = 2 - \frac{2}{n}$ is provided with the L^1-preserving self-similar transformation

$$u_\mu(x,t) = \mu^n u(\mu x, \mu^n t),$$

see §2.3.1. This transformation induces the backward self-similar transformation

$$v(y,s) = (T-t)u(x,t)$$
$$y = (x - x_0)/(T-t)^{1/n}$$
$$s = -\log(T-t)$$

and the rescaled equation

$$\begin{aligned} v_s &= \frac{m-1}{m}\Delta v^m - \nabla\cdot v\nabla(\Gamma * v + \frac{|y|^2}{2n}) \\ v &\geq 0 \qquad\qquad \text{in } \mathbf{R}^n \times (-\log T, +\infty). \end{aligned}$$

The constant $\lambda_* = \lambda_*(n) > 0$ defined by (2.139) is the best constant of Wang-Ye's dual form of the Trudinger-Moser inequality, see,[311,331]

$$\inf\left\{\mathscr{F}(u) \mid 0 \leq u \in L^m(\mathbf{R}^n),\ \int_{\mathbf{R}^n} u = \lambda_*\right\} > -\infty$$

with the left-hand side equal to 0, where $m = 2 - \frac{2}{n}$.

2.2.7. Summary

We have studied the particle kinetics, particularly, stationary mean field turbulence and the molecular kinetics using equilibrium statistical mechanics, Kramers-Moyal expansion, and kinetic theory.

(1) Two-dimensional vortex point system is described by a Hamilton system.
(2) The equilibrium statistical mechanics formulates the probability measure of ensembles, which are equivalent classes of the microscopic states that have the same macroscopic profiles using the principle of equal *a priori* probabilities. Then, the ergodic hypothesis asserts its practical validity.
(3) Micro-canonical, canonical, and grand-canonical ensembles are defined for materially and kinetically closed, materially closed, and open systems, respectively. In connection with the non-equilibrium thermodynamics, the micro-canonical setting is concerned with the total energy conservation and the entropy increasing, the canonical setting is concerned with the temperature conservation and Helmholtz' free energy decreasing, and finally, the grand-canonical setting is concerned with the pressure conservation and Gibbs' free energy decreasing.
(4) We obtain the mean field equation of high-energy limit of two- dimensional vortex systems by several arguments. This equation is provided with the dual variation between the vortex point distribution and the stream function. Using the latter, it is described by the elliptic eigenvalue problem with the exponential nonlinearity competing two-dimensional diffusion, whereby the quantized blowup mechanism is observed.
(5) The particle equation of the system of chemotaxis is associated with the molecular transport, and non-equilibrium mean field equations are derived from the master equation.
(6) There is an equivalent mezoscopic approach using several entropies to describe the motion of the mean field, called the kinetic theory. Mass quantization is observed for the equation derived from the canonical setting and Rényi-Tsallis' entropy.

2.3. Gauge Field

The first quantization in quantum mechanics is an operation for describing the movement of the particle density from that of classical particles. If we take this process, using the gauge of the Maxwell equation, then the Bogmol'nyi structure and the self-duality induce again the exponential nonlinearity competing to the two-dimensional diffusion. This section is devoted to the quantized blowup mechanism for solutions to the *mean field equation* in two-space dimension. We develop the blowup analysis based on the scaling invariance of the problem and reveals the quantized blowup mechanism through several hierarchical arguments, and, consequently, support the motivation of §1.1 from the theory of nonlinear spectral mechanics.

2.3.1. Field Theory

In this paragraph, we describe fundamental notions of gauge theory, particularly, covariant derivative and self-duality, and derive the gauged Schrödinger Chern-Simons equation.[317,318]

Classical Mechanics

First, if $q_1,\dots,q_N$ are particles in $\mathbf{R}^3$ with the masses $m_1,\dots,m_N$, then the Newton equation is described by

$$m_i\ddot{q}_i = \sum_{j\neq i}\frac{\partial}{\partial q_j}\left(-\frac{m_i m_j}{|q_i-q_j|}\right), \qquad i=1,\dots,N. \tag{2.141}$$

This equation is equivalent to

$$\delta\int L = 0, \tag{2.142}$$

using the Lagrangian

$$L = \sum_i \frac{1}{2}m_i\dot{q}_i^2 - \sum_{j<i}\frac{m_i m_j}{|q_i-q_j|},$$

that is (2.141) is equivalent to the Euler-Lagrange equation associated with this L and described by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \qquad i=1,\dots,N. \tag{2.143}$$

Then, Hamilton's canonical equation (2.81) is obtained by using the general momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - L,$$

regarded as $H = H(q_1,\dots,q_N;p_1,\dots,p_N)$.

Gauge

Motion of the electron put in the electro-magnetic field is described by

$$m\ddot{x} = -Q(E + \dot{x} \times B) - \nabla V,$$

using the position $x = x(t) \in \mathbf{R}^3$, mass $m > 0$, electric charge Q, electric field E, magnetic field B, and kinetic potential V. Actually, QE and $Q\dot{x} \times B$ denote the electric field action force and the Lorentz force, respectively. These electric and magetic fields are subject to Maxwell's equation

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0, \end{aligned}$$

and, therefore,

$$\begin{aligned} B &= -\nabla \times A \\ E &= -\nabla \Psi + \frac{\partial A}{\partial t} \end{aligned}$$

using the vector and the scalar potentials A and Ψ, respectively.
Writing $A = (A_i)_{i=1,2,3}$ and $y = m\dot{x} = (y_i)_{i=1,2,3}$, it holds that

$$\begin{aligned} \dot{y}_i &= -Q(E + \dot{x} \times B)_i - \frac{\partial V}{\partial x^i} \\ &= Q\left(\frac{\partial \Psi}{\partial x^i} - \frac{\partial A_i}{\partial t}\right) + Q\dot{x}^j\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) - \frac{\partial V}{\partial x^i} \\ &= -Q\frac{dA_i}{dt} + Q\frac{\partial \Psi}{\partial x^i} + Q\dot{x}^j\frac{\partial A_j}{\partial x^i} - \frac{\partial V}{\partial x^i}, \end{aligned}$$

and hence we obtain

$$\frac{d}{dt}(y_i + QA_i) = \frac{\partial}{\partial x_i}\left(Q\Psi + Q\sum_j \dot{x}^j A_j - V\right).$$

This equation means (2.143) for

$$L = \frac{1}{2}m\dot{x}^2 + Q\Psi + Q\dot{x} \cdot A - V$$

and $N = 3$. Thus the general momentum and the Hamiltonian are described by

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = y_i + QA_i$$

and

$$\begin{aligned} H &= \sum_i p_i \dot{x}^i - L \\ &= \frac{1}{m}(y_i + QA_i)y_i - \left\{ \frac{1}{2m} y_i^2 + Q\Psi + \frac{1}{m} Q y_i A_i - V \right\} \\ &= \frac{1}{2m} y^2 - Q\Psi + V \\ &= \frac{1}{2m}(p_i - QA_i)^2 - Q\Psi + V, \end{aligned}$$

respectively. Regarding $q = x$ as the general coordinate and putting $\Psi = A_0$, we reformulate the Hamiltonian of this system by

$$H = H(q,p,t) = \frac{1}{2m}(p_i - QA_i)^2 - QA_0 + V.$$

First Quantization

The particle mass is quantized by the Schrödinger equation. In the first quantization, the electro-magnetic field remains classical, coupled with the vector potential A_μ, $\mu = 0,1,2,3$. Thus we replace $E = H(q,p,t)$ by

$$\begin{aligned} t &\mapsto t \\ E &\mapsto \imath \frac{h}{2\pi} \frac{\partial}{\partial t} \\ p_i &\mapsto \frac{1}{\imath} \frac{h}{2\pi}. \end{aligned}$$

Using the normalization of $\frac{h}{2\pi} = 1$, we obtain

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2m}(\partial_i - \imath Q A_i)^2 \psi - QA_0 \psi + V\psi, \tag{2.144}$$

and then the gauge covariant derivatives

$$D_\mu = \partial_\mu - \imath Q A_\mu, \qquad \mu = 0,1,2,3,$$

replace (2.144) by

$$\imath D_0 \psi = -\frac{1}{2m} D_i^2 \psi + V\psi, \tag{2.145}$$

where $\partial_0 = \partial_t$.
The energy operator is thus described by

$$\hat{E} = -\frac{1}{2m} D_i^2 + (V - QA_0)$$

and, therefore, we obtain the energy expectation,

$$\begin{aligned} E &= \int_{\mathbf{R}^3} \overline{\psi} \hat{E} \psi \\ &= \int_{\mathbf{R}^3} \overline{\psi} \left\{ -\frac{1}{2m} D_i^2 \psi + (V - QA_0) \psi \right\} \\ &= \int_{\mathbf{R}^3} \frac{1}{2m} |D_i \psi|^2 + (V - QA_0) |\psi|^2 \, dx \end{aligned}$$

by

$$\begin{aligned} \partial_i (\overline{\psi} D_i \psi) &= \partial_i \overline{\psi} D_i \psi + \overline{\psi} \partial_i D_i \psi \\ &= \overline{D_i \psi} D_i \psi + \overline{\psi} D_i D_i \psi \\ &= |D_i \psi|^2 + \overline{\psi} D_i^2 \psi. \end{aligned}$$

This E is regarded as a Hamiltonian again, denoted by H, and then (2.144) is equivalent to

$$\imath \frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \overline{\psi}}.$$

The equivalent formulation (2.145) is obtained by replacing H and ∂_t by

$$H = \int_{\mathbf{R}^3} \frac{1}{2m} |D_i \psi|^2 + V |\psi|^2 \, dx$$

and the covariant derivative D_0, respectively, that is

$$\imath D_0 \psi = \frac{\delta H}{\delta \overline{\psi}}.$$

Chern-Simons Relation

The pre-gauged Schrödinger equation is defined by

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta \psi + V \psi,$$

and then there arises the nonlinear Schrödinger equation

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta \psi - g |\psi|^2 \psi, \tag{2.146}$$

if the potential $V = -g|\psi|^2$ is adopted to describe the mean field self-interaction, where g denotes the gravitational constant. Henceforth, we consider the case of two space dimension. The particle density $\rho = |\psi|^2 = \psi \overline{\psi}$ is subject to

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \psi \frac{\partial \overline{\psi}}{\partial t} + \overline{\psi} \frac{\partial \psi}{\partial t} \\ &= -\frac{\imath}{2m} (\psi \Delta \overline{\psi} - \overline{\psi} \Delta \psi) \\ &= -\frac{\imath}{2m} \partial_k (\psi \partial_k \overline{\psi} - \overline{\psi} \partial_k \psi) \end{aligned}$$

or, equivalently,

$$\rho_t + \nabla \cdot j = 0$$

for

$$\begin{aligned} j^k &= \frac{\imath}{2m}(\psi \partial_k \overline{\psi} - \overline{\psi} \partial_k \psi), \ \ k = 1,2 \\ j &= (j^k)_{k=1,2} \end{aligned}$$

which results in the Maxwell equation

$$\partial_\mu J^\mu = 0, \tag{2.147}$$

where

$$J = (J^\mu) = (\rho, j^k) = (\rho, j), \qquad \mu = 0,1,2$$

denotes the current density.
Equation (2.146) is described by (2.142) for

$$L = \imath \overline{\psi} \partial_0 \psi - \frac{1}{2m} |\partial_k \psi|^2 + \frac{g}{2} |\psi|^4,$$

and we obtain the gauged nonlinear Schrödinger equation

$$\imath D_0 \psi = -\frac{1}{2m} D_k^2 \psi - g |\psi|^2 \psi, \tag{2.148}$$

using the gauge covariant derivative $D_\mu = \partial_\mu - \imath Q A_\mu$. The current density, on the other hand, is equal to

$$\begin{aligned} J &= (J^\mu) = (\rho, J^k) \\ \rho &= |\psi|^2 \\ J^k &= \frac{\imath}{2m}(\psi \overline{D_k \psi} - \overline{\psi} D_k \psi), \ \ k = 1,2 \end{aligned}$$

by

$$\partial_\mu (\psi_1 \overline{\psi_2}) = \psi_1 \overline{D_\mu \psi_2} + (D_\mu \psi_1) \overline{\psi_2}, \qquad \mu = 0,1,2.$$

Since this J is subject to (2.147), we take the gauge potential A_μ satisfying

$$\partial_\nu F_{\mu\nu} = -J^\mu, \qquad \mu = 0,1,2,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the rate of change of electromagnetic field. In the Chern-Simons theory, however, this relation is replaced by

$$F_{\mu\nu} = \frac{1}{\kappa} \varepsilon_{\mu\nu\alpha} J^\alpha, \qquad \mu, \nu, \alpha = 0,1,2, \tag{2.149}$$

using the coupling constant $\kappa > 0$ and the skew-symmetric tensor $\varepsilon_{\mu\nu\alpha}$ normalized by $\varepsilon_{012} = 1$.

Thus (2.148) with (2.149) comprises of the gauged Schrödinger-Chern-Simons equation, where

$$\begin{aligned}
&J = (J^{\mu}) \\
&J^0 = |\psi|^2 \\
&J^k = \frac{\iota}{2m}(\psi\overline{D_k\psi} - \overline{\psi}D_k\psi), \;\; k = 1,2 \\
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\end{aligned}$$

and this equation is described by (2.142) for

$$\begin{aligned}
L &= -\frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha + \iota\overline{\psi}D_0\psi - \frac{1}{2m}|D_k\psi|^2 + \frac{g}{2}|\psi|^4 \\
&= -\frac{\kappa}{4}\varepsilon\ \mu\nu\alpha A_\mu F_{\nu\alpha} + \iota\overline{\psi}D_0\psi - \frac{1}{2m}|D_k\psi|^2 + \frac{g}{2}|\psi|^4.
\end{aligned}$$

Once the Lagrangian action density is defined, then the Bogomol'nyi structure and self-duality induce a system of first order equations and a single second order equation, respectively, see below.

Bogomol'nyi Equation

In two-space dimension, the *Lagrangian action density* concerning super-conductivity is associated with the carrier (A_0, A_1, A_2) and the Higgs field φ, where A_0 and A_i $(i = 1,2)$ are the electric and magnetic gauge fields, respectively, and $|\varphi|^2$ describes the Cooper pair density. In the law temperature of $A_0 = 0$, non-relativistic Ginzburg-Landau density and the retivistic Abelian-Higgs density share the same stationary state involved by the Higgs term and the double-well potential,[124,225] that is

$$L(A,\varphi) = \int_{\mathbf{R}^2} \frac{1}{2}\left|(\partial_j - \iota A_j)\varphi\right|^2 + \frac{1}{4}F_{jk}F_{jk} + \frac{\lambda}{8}\left(|\varphi|^2 - 1\right)^2\, dx.$$

Here, the Higgs field $\varphi : \mathbf{R}^2 \to \mathbf{C}$ is regarded as a cross-section of a $\mathbf{C}$-line bundle on $\mathbf{R}^2$, $A = (A_j)_{j=1,2}$ is the connection, and

$$F_{jk} = \partial_j A_k - \partial_k A_j$$

is its curvature tensor. From these topological requirements, it holds that

$$n = \frac{1}{2\pi}\int_{\mathbf{R}^2} F_{12} \in \mathbf{Z}$$

under the assumption of

$$\begin{aligned}
&|\varphi| \to 1 \\
&D_A^j\varphi = \partial_j\varphi - \iota A_j\varphi \to 0 \qquad \text{as } |x| \to \infty,
\end{aligned} \tag{2.150}$$

and this n is called the vortex number. It is actually the first Chern class of the $\mathbf{C}$-line bundle. If $A = 0$, then the above $L(A,\varphi)$ describes the Landau-Ginzburg vortices.[22]

Putting $\varphi = \varphi_1 + \imath\varphi_2$, we have

$$\begin{aligned}\frac{1}{2}\left|(\partial_j - \imath A_j)\varphi\right|^2 &= \frac{1}{2}\left\{(\partial_1\varphi_1 + A_1\varphi_2)^2 + (\partial_1\varphi_2 - A_1\varphi_1)^2\right\}\\ &\quad + \frac{1}{2}\left\{(\partial_2\varphi_1 + A_2\varphi_2)^2 + (\partial_2\varphi_2 - A_2\varphi_1)^2\right\}\\ &= \frac{1}{2}\left[(\partial_1\varphi_1 + A_1\varphi_2) - (\partial_2\varphi_2 - A_2\varphi_1)\right]^2\\ &\quad + \frac{1}{2}\left[(\partial_2\varphi_1 + A_2\varphi_2) + (\partial_1\varphi_2 - A_1\varphi_1)\right]^2\\ &\quad + \frac{1}{2}(A_1\partial_2 - A_2\partial_1)(\varphi_1^2 + \varphi_2^2)\end{aligned}$$

and

$$\frac{1}{2}\int_{\mathbf{R}^2}(A_1\partial_2 - A_2\partial_1)(\varphi_1^2 + \varphi_2^2) = \frac{1}{2}\int_{\mathbf{R}^2} F_{12}(\varphi_1^2 + \varphi_2^2).$$

In the case of $\lambda = 1$, therefore, it holds that

$$\begin{aligned}L(A,\varphi) = \int_{\mathbf{R}^2} &\frac{1}{2}\left[(\partial_1\varphi_1 + A_1\varphi_2) - (\partial_2\varphi_2 - A_2\varphi_1)\right]^2\\ &+ \frac{1}{2}\left[(\partial_2\varphi_1 + A_2\varphi_2) + (\partial_1\varphi_2 - A_1\varphi_1)\right]^2\\ &+ \frac{1}{2}\left[F_{12} + \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 - 1\right)\right]^2 dx + \frac{1}{2}\int_{\mathbf{R}^2} F_{12}\end{aligned}$$

by

$$\begin{aligned}&\frac{1}{2}\left[F_{12} + \frac{1}{2}(\varphi_1^2 + \varphi_2^2 - 1)\right]^2\\ &\quad = \frac{1}{2}F_{12}^2 + \frac{1}{8}\left(|\varphi|^2 - 1\right)^2 + \frac{1}{2}F_{12}(\varphi_1^2 + \varphi_2^2 - 1).\end{aligned}$$

Thus we obtain $L \geq n\pi$ with the equality if and only if

$$\begin{aligned}&(\partial_1\varphi_1 + A_1\varphi_2) - (\partial_2\varphi_2 - A_2\varphi_1) = 0\\ &(\partial_2\varphi_1 + A_2\varphi_2) + (\partial_1\varphi_2 - A_1\varphi_1) = 0\\ &F_{12} + \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 - 1\right) = 0.\end{aligned} \tag{2.151}$$

Self-Duality

The above described Bogomol'nyi equation[27] is reduced to a single second order nonlinear equation, similarly to the axially symmetric four dimensional self-dual $SU(2)$ Yang-Mills equation.[317,318,337]

More precisely, the first two relations of (2.151) are regarded as the real and the imaginary parts of

$$2\overline{\partial}\varphi - \imath\hat{A}\varphi = 0,$$

where

$$\hat{A} = A_1 + \imath A_2$$
$$\partial = \frac{1}{2}(\partial_1 - \imath\partial_2)$$
$$\overline{\partial} = \frac{1}{2}(\partial_1 + \imath\partial_2).$$

This formulation implies $\hat{A} = -2\imath\overline{\partial}\log\varphi$, and in particular,

$$\varphi = e^f$$

for some $f = f_1 + \imath f_2$ with

$$f_2(\theta, r) \equiv f_2(\theta + 2\pi, r) \qquad \text{modulo } 2\pi\mathbf{Z}. \tag{2.152}$$

Then, we obtain

$$A_1 = \partial_2 f_1 + \partial_2 f_1$$
$$A_2 = -\partial_1 f_1 + \partial_2 f_2,$$

and, therefore,

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = -\Delta f_1.$$

It holds that

$$-\Delta f_1 + \frac{1}{2}\left(e^{2f_1} - 1\right) = 0$$

and

$$f_1 \to 0, \quad \varphi \to e^{\imath f_2} \qquad \text{as } |x| \to \infty$$

by (2.150)-(2.151).
If f_2 has a finite number of singular points denoted by

$$\{a_1, \ldots, a_m\} \subset \mathbf{R}^2$$

then $\varphi(a_k) = 0$ because $\varphi(x)$ is continuous. If n_k denotes the order of zero, more precisely, then

$$f_1(x) \sim \frac{n_k}{2}\log|x - a_k|^2 \qquad \text{as } x \to a_k.$$

Thus writing

$$v = 2f_1 - \sum_k n_k \log|x - a_k|^2,$$

we obtain

$$\Delta v = e^v - 1 + 4\pi\sum_k n_k \delta(x - a_k) \qquad \text{in } \mathbf{R}^2.$$

Several elliptic equations involving exponential nonlinearity are obtained in stationary self-dual gauge theories in the two-space dimension by similar arguments,[348] that is non-relativistic and relativistic super-conductivities in high temperature are described by the gauged Schrödinger and the Chern-Simons densities with the self-dual equations defined by

$$\Delta v = e^v + 4\pi \sum_k n_k \delta(x - a_k) \qquad \text{in } \mathbf{R}^2$$

and

$$\Delta v = e^v(e^v - 1) + 4\pi \sum_k n_k \delta(x - a_k) \qquad \text{in } \mathbf{R}^2, \tag{2.153}$$

respectively. The effect of vortex term $4\pi \sum_k \delta(x - a_k)$ is particularly studied in detail.[316] Finally, the doubly periodic second solution to (2.153) without vortices is reduced to the mean field equation

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \qquad \text{in } \Omega$$
$$\int_\Omega v = 0, \tag{2.154}$$

where $\Omega = \mathbf{R}^2 / a\mathbf{Z} \times b\mathbf{Z}$, $a, b > 0$ and $\lambda = 4\pi$, see.[315]

2.3.2. Exponential Nonlinearity Revisited

As we have seen, the elliptic eigenvalue problem

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{2.155}$$

relative to (2.154) describes stationary mean field of many vortices in the perfect fluid, stationary self-dual gauge field associated with super-conductivity, and stationary state of the chemotaxis system, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a constant, and $v = v(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution. For this problem, several results have been obtained such as the classification of the singular limit, uniqueness of the solution, singular perturbation, and the topological degree calculation.

Theorem 2.7 ([213]). *If (λ_k, v_k) is a solution to (2.155) for*

$$(\lambda, v) = (\lambda_k, v_k), \qquad k = 1, 2, \dots$$

and

$$\lambda_k \to \lambda_0 \in [0, \infty)$$
$$\|v_k\|_\infty \to +\infty \tag{2.156}$$

as $k \to \infty$, then $\lambda_0 = 8\pi m$ for some $m \in \mathbf{N}$, and there is $\{v_k'\} \subset \{v_k\}$ such that

$$v_k' \to 8\pi \sum_{k=1}^m G(\cdot, x_k^*) \tag{2.157}$$

locally uniformly in $\overline{\Omega} \setminus \mathscr{S}$, where $\mathscr{S}$ is the blowup set of $\{v_k'\}$ defined by

$$\mathscr{S} = \left\{ x_0 \in \overline{\Omega} \mid \text{there is } x_k' \to x_0 \text{ such that } v_k'(x_k') \to +\infty \right\}.$$

This blowup set is composed of m-distict interior points denoted by

$$\mathscr{S} = \{x_1^*, \dots, x_m^*\} \subset \Omega,$$

satisfying

$$\frac{1}{2}\nabla R(x_j^*) + \sum_{k \neq j} \nabla_x G(x_j^*, x_k^*) = 0, \qquad j = 1, \dots, m, \tag{2.158}$$

where $G = G(x,x')$ denotes the Green's function of $-\Delta_D$ defined by

$$-\Delta_x G(\cdot, x') = \delta(\cdot - x') \quad \text{in } \Omega, \qquad G(\cdot, x') = 0 \quad \text{on } \partial\Omega$$

for $x' \in \Omega$ and

$$R(x) = \left[G(x,x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}, \qquad x \in \Omega$$

is the Robin function.

We have

$$\frac{e^{v'_k}}{\int_\Omega e^{v'_k}} dx \rightharpoonup 8\pi \sum_{j=1}^m \delta_{x_j^*}(dx)$$

in $\mathscr{M}(\overline{\Omega})$, and furthermore, (2.158) is equivalent for

$$(x_1^*, \dots, x_m^*) \in \Omega \times \dots \times \Omega$$

to be a critical point of the Hamiltonian

$$H(x_1, \dots, x_m) = \frac{1}{2} \sum_{j=1}^m R(x_j) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1; j \neq i}^m G(x_i, x_j).$$

This fact means a recursive profile of the vortex mean field hierarchy, see (2.79). Besides the stationary state of chemotaxis (1.25), the above described quantized blowup mechanism is actually observed in several problems. The quantized mechanism, conversely, clarifies the structure of the total set of solutions mathematically, and we obtain the following theorems.

Theorem 2.8 ([301,308]). *If $0 < \lambda < 8\pi$, then there is a unique solution to (2.155).*

Theorem 2.9 ([15]). *If $(x_1^*, \dots, x_m^*) \in \Omega \times \dots \times \Omega$ is a non-degenerate critical point of $H(x_1, \dots, x_m)$, then there is a family of solutions $\{(\lambda_k, v_k)\}_k$ to (2.155) satisfying (2.156) for $\lambda_0 = 8\pi m$ and (2.157) to $v'_k = v_k$ locally uniformly in $\overline{\Omega} \setminus \mathscr{S}$.*

Theorem 2.10 ([54,55]). *Total topological degree of (2.155), denoted by $d(\lambda)$, is determined by the genus g of Ω if $\lambda \notin 8\pi\mathbf{N}$. More precisely, it holds that*

$$d(\lambda) = \binom{m+1-g}{m}$$

for $8\pi m < \lambda < 8\pi(m+1)$. In particular, if $g \geq 1$ and $\lambda \notin 8\pi\mathbf{N}$, then (2.155) admits a solution

Collapse Collision

The above described quantized blowup mechanism is still vaild without the boundary condition or even to variable coefficients,

$$-\Delta v = V(x)e^{v} \qquad \text{in } \Omega, \tag{2.159}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain. If $v = v(x)$ is a solution to (2.155), then $w = v + \log\lambda - \log\int_\Omega e^v$ solves

$$-\Delta w = e^{w} \qquad \text{in } \Omega$$
$$\int_\Omega e^{w} = \lambda.$$

The following theorem and its proof are, therefore, quite useful in the study of the self-dual gauge field equation with vortex terms. There may arise, however, the collision of collapses, that is multiple bubbles. Thus in Theorem 2.7, we exclude the collision of the blowup points and also prescribe the location of blowup points, provided with the boundary condition.

Theorem 2.11 ([181]). *Let $v_k = v_k(x)$, $k = 1, 2, \ldots$ satisfy*

$$-\Delta v_k = V_k(x)e^{v_k}$$
$$0 \le V_k(x) \le C_1 \qquad \text{in } \Omega$$
$$\int_\Omega e^{v_k} \le C_2,$$

where $C_1, C_2 > 0$ are constants, $V_k = V_k(x)$ is continuous, and $V_k \to V$ uniformly on $\overline{\Omega}$. Then, passing to a sub-sequence, we have the following alternatives:

(i) *$\{v_k\}$ is locally uniformly bounded in Ω.*
(ii) *$v_k \to -\infty$ locally uniformly in Ω.*
(iii) *We have a finite set $\mathscr{S} = \{a_i\} \subset \Omega$ and $m_i \in \mathbf{N}$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathscr{S}$ and*

$$V_k(x)e^{v_k}dx \rightharpoonup \sum_i 8\pi m_i \delta_{a_i}(dx)$$

in $\mathscr{M}(\Omega)$. Here, $\mathscr{S}$ is the blowup set of $\{v_k\}$ in Ω.

For the proof, we use the preliminary version sometimes called the rough estimate. It is stated as follows.

Theorem 2.12 ([32]). *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, and $v_k = v_k(x)$, $k = 1, 2, \ldots$ satisfy*

$$-\Delta v_k = V_k(x)e^{v_k} \qquad \text{in } \Omega$$

for $V_k = V_k(x) \geq 0$, *and assume the existence* $c_1, c_2 > 0$ *such that*

$$\|V_k\|_\infty \leq c_1$$
$$\|e^{v_k}\|_1 \leq c_2$$

for $k = 1, 2, \ldots$. *Then, passing to a sub-sequence, we have the following alternatives:*

(i) $\{v_k\}$ *is locally uniformly bounded in* Ω.
(ii) $v_k \to -\infty$ *locally uniformly in* Ω.
(iii) *There is a finite set* $\mathscr{S} = \{a_i\} \subset \Omega$ *and* $\alpha_i \geq 4\pi$ *such that* $v_k \to -\infty$ *locally uniformly in* $\Omega \setminus \mathscr{S}$ *and*

$$V_k(x)e^{v_k}dx \rightharpoonup \sum_i \alpha_i \delta_{a_i}(dx)$$

in $\mathscr{M}(\Omega)$. *Furthermore,* $\mathscr{S}$ *is the blowup set of* $\{v_k\}$ *in* Ω.

Thanks to Theorem 2.12, the proof of Theorem 2.11 is reduced to the following case, where $B = B(0,R) \subset \mathbf{R}^2$ and $B_r = B(0,r)$.

Theorem 2.13. *If*

$$\begin{aligned}
&-\Delta v_k = V_k(x)e^{v_k}, \ \ V_k(x) \geq 0 && \text{in } B \\
&V_k \to V && \text{in } C(\overline{B}) \\
&\max_{\overline{B}} v_k \to +\infty && \\
&\max_{\overline{B}\setminus B_r} v_k \to -\infty, && 0 < r < R \\
&\lim_{k\to\infty} \int_B V_k e^{v_k} = \alpha && \\
&\int_B e^{v_k} \leq C_0, &&
\end{aligned}$$

then it holds that $\alpha = 8\pi m$ *for some* $m \in \mathbf{N}$.

We can use this quantization to construct the solution for the disquantized λ, see.[180,297] The origin of this property is the self-similarity of (2.159) described below. If the boundary condition is provided, then the mutiple blowup, indicated by $m_i \geq 2$, cannot occur,[180,216,235] while there is acutal collision in the general case.[57] The proof of Theorem 2.11 says that the non-compact solution sequence is approximated by a finite sum of rescaled entire solutions. The required estimate of mass from below is standard, using the self-similarity and the classification of the entire solution. The estimate from above or the *residual vanishing* is, thus, essential. It is obtained by the sup + inf inequality formulated by Theorem 2.22 below, combined with the scaling argument.
If the boundary condition is provided, this process can be replaced by the profile of the asymptotic symmetry of the solution[180] or the method of the second moment[216,235] . In

more precise, we obtain $m_i = 1$ in Theorem 2.11, provided that

$$\max_{\partial\Omega} v_k - \min_{\partial\Omega} v_k \leq C$$
$$\|\nabla V_k\|_\infty \leq C \tag{2.160}$$

for $k = 1, 2, \ldots$. Similarly, if (2.160) holds for $\Omega = B$, then it follows that $\alpha = 8\pi$ and

$$\left| v_k(x) - \log \frac{e^{v_k(0)}}{\left(1 + \frac{V_k(0)}{8} e^{v_k(0)} |x|^2\right)^2} \right| \leq C$$

for $x \in B$ and $k = 1, 2, \ldots$ in Theorem 2.13.
Theorem 2.11, however, has a variety of implications because it is free from any boundary condition. For instance, even if the original problem is provided with the boundary condition, the rescaled problem loses it. It often happens that this fact causes a trouble in the blowup analysis.

2.3.3. Scaling Invariance

Equation (2.271) or its general form (2.159) is scaling invariant, that is it is invariant under the transformation

$$v^\mu(x) = v(\mu x) + 2\log\mu$$
$$V^\mu(x) = V(\mu x) \tag{2.161}$$

for $\mu > 0$. Such a property, called *scaling invariance* or *self-similarity*, is observed in several nonlinear partial differential equations and is used essentially for their mathematical analysis as we have seen in §1.1.
A typical example is

$$-\Delta v = v^p, \quad v > 0$$

for $1 < p < \infty$. Thus if $v = v(x)$ is a solution to this equation, then so is $v_\mu(x) = \mu^{2/(p-1)} v(\mu x)$ for $\mu > 0$. Similarly, if $v = v(x,t)$ is a solution to

$$v_t - \Delta v = v^p, \quad v > 0, \tag{2.162}$$

then so is

$$v_\mu(x,t) = \mu^{2/(p-1)} v(\mu x, \mu^2 t). \tag{2.163}$$

This structure causes obstruction to construct (weak) solution, while there are applications of renormalization group and blowup analysis. In the blowup analysis, we classify non-compact (approximate) solutions and obtain actual solution by excluding these possibilities. Applications of such argument to the calculus of variation are described in Bahri,[12] and for example,

$$-\Delta v = v^5, \; v > 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

has a solution if the bounded domain $\Omega \subset \mathbf{R}^3$ is not contractible, see §2.1.2. Another application is the proof of the *a priori* bound of the solution to the sub-critical elliptic equation, which guarantees the topological degree approach to this problem.[73,117]

Theorem 2.14. *Let $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $1 < p < \frac{n+2}{n-2}$. Then, there is $C > 0$ such that any solution $v = v(x)$ to*

$$-\Delta v = v^p, \ \ v > 0 \quad in\ \Omega, \qquad v = 0 \quad on\ \partial\Omega \tag{2.164}$$

satisfies

$$\|v\|_\infty \leq C. \tag{2.165}$$

Here, we describe the outline of the proof to illustrate the fundamental concept of the blowup analysis. In fact, first, we assume the contrary, the existence of the sequence of the solutions to (2.164), denoted by $\{v_k\}$, satisfying

$$v_k(x_k) = \|v_k\|_\infty \to +\infty$$

for $x_k \in \Omega$. Then, we take

$$\tilde{v}_k(x) = \mu_k^{2/(p-1)} v_k(\mu_k x + x_k)$$

for $\mu_k = v_k(x_k)^{-(p-1)/2} \to 0$ and obtain

$$\begin{aligned} &-\Delta \tilde{v}_k = \tilde{v}_k^p, \ \ \tilde{v}_k > 0 \quad \text{in } \Omega_k, \qquad \tilde{v}_k = 0 \quad \text{on } \partial\Omega_k \\ &\tilde{v}_k(0) = \max_x \tilde{v}_k(x) = 1, \end{aligned}$$

where $\Omega_k = \mu_k^{-1}(\Omega - \{x_k\})$. From the boundary condition and the elliptic estimate, we obtain a subsequence, denoted by the same symbol such that $\tilde{v}_k \to v$ locally uniformly in $\mathbf{R}^n$ for a smooth $v = v(x)$, and it follows that either

$$-\Delta v = v^p, \quad 1 = v(0) \geq v \geq 0 \qquad \text{in } \mathbf{R}^n \tag{2.166}$$

or

$$\begin{aligned} &-\Delta v = v^p, \quad 1 = v(0) \geq v \geq 0 \qquad \text{in } \mathbf{R}^n_+ \\ &v = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\mathbf{R}^n_+, \end{aligned} \tag{2.167}$$

where $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_n > 0\}$ denotes the half space.
Problem (2.166), however, admits no solution in the case of $1 < p < \frac{n+2}{n-2}$. The same is true for (2.167) if $1 < p \leq \frac{n+2}{n-2}$, and hence we end up with (2.165).
Ingredients of the blowup analysis are thus summarized as follows:

(1) scaling invariance.
(2) classification of the entire solution.
(3) control at infinity of the rescaled solution.
(4) hierarchical argument.

In the case of (2.155), first, we use complex-geometric structure and linear theory associated with the real analysis. This study comprises of the pre-scaled analysis for the proof of Theorem 2.12, see §§2.3.4, 2.3.5, and 2.3.6. Next, method of the moving plane is applied to classify the entire solution, see §2.3.7. Finally, control at infinity of the rescaled solution is done by the Harnack type inequality. We thus complete the proof of Theorem 2.13 in §2.3.8 by the hierarchical argument using (2.161).
The self-similarity (2.163) induces forward and the backward self-similar transformations to (2.162), see §2.163. The underlying structure of these transformations takes a fundamental role in the study of the system of chemotaxis, see §1.1.

2.3.4. Liouville-Bandle Theory

The problem (2.155) is associated with the complex function theory and the theory of surfaces. In this paragraph, we describe them briefly.

Complex Structure

If we identify $x=(x_1,x_2)\in\Omega$ to $z=x_1+\imath x_2\in\mathbf{C}$, then

$$-\Delta u = e^u \tag{2.168}$$

means

$$u_{z\bar{z}} = -\frac{1}{4}e^u$$

for $\bar{z}=x_1-\imath x_2$. This implies

$$\begin{aligned} s_{\bar{z}} &= u_{zz\bar{z}} - u_z u_{z\bar{z}} \\ &= -\frac{1}{4}e^u u_z + \frac{1}{4}u_z e^u = 0 \end{aligned}$$

for

$$s = u_{zz} - \frac{1}{2}u_z^2, \tag{2.169}$$

and, therefore, $s=s(z)$ is a holomorphic function of $z\in\Omega\subset\mathbf{C}$.
Regarding (2.169) as a Riccati equation of u, we obtain

$$\varphi_{zz} + \frac{1}{2}s\varphi = 0 \tag{2.170}$$

for $\varphi = e^{-u/2}$. Here, we take $x^*=(x_1^*,x_2^*)\in\Omega$, and define the fundamental system of the linear equation (2.170), denoted by

$$\{\varphi_1(z),\varphi_2(z)\},$$

such that

$$\begin{aligned} \varphi_1|_{z=z^*} &= \left.\frac{\partial\varphi_2}{\partial z}\right|_{z=z^*} = 1 \\ \left.\frac{\partial\varphi_1}{\partial z}\right|_{z=z^*} &= \varphi_2|_{z=z^*} = 0, \end{aligned} \tag{2.171}$$

where $z^* = x_1^* + \imath x_2^*$. This $\{\varphi_1(z), \varphi_2(z)\}$ forms a system of analytic functions of $z \in \Omega$, and it holds that

$$\varphi = e^{-u/2} = \overline{f_1}(\overline{z})\varphi_1(z) + \overline{f_2}(\overline{z})\varphi_2(z) \tag{2.172}$$

with some functions $\overline{f_1}$ and $\overline{f_2}$ of $\overline{z}$.
These functions are prescribed by the Wronskian. It holds that

$$W(\varphi_1, \varphi_2) \equiv \varphi_1\varphi_{2z} - \varphi_{1z}\varphi_2 = 1,$$

and hence

$$\begin{aligned}\overline{f_1}(\overline{z}) &= W(\varphi, \varphi_2) = \varphi\varphi_{2z} - \varphi_z\varphi_2\\ \overline{f_2}(\overline{z}) &= W(\varphi_1, \varphi) = \varphi_1\varphi_z - \varphi_{1z}\varphi.\end{aligned}$$

Here, the left-hand side is independent of z. Taking $z = z^*$ in the right-hand side, we obtain

$$\begin{aligned}\overline{f_1}(\overline{z}) &= \varphi(z^*, \overline{z})\\ \overline{f_2}(\overline{z}) &= \varphi_z(z^*, \overline{z}).\end{aligned}$$

Since φ is real-valued, there holds that

$$\varphi_{\overline{z}\overline{z}} + \frac{1}{2}\overline{s}\varphi = 0$$

for $\overline{s} = \overline{s}(\overline{z})$ defined by $\overline{s}(\overline{z}) = \overline{s(z)}$. Thus this equality is valid to $\overline{f_1}(\overline{z})$ and $\overline{f_2}(\overline{z})$, while $\{\overline{\varphi_1}, \overline{\varphi_2}\}$ forms a fundamental system such that

$$\begin{aligned}\overline{\varphi_1}|_{\overline{z}=\overline{z^*}} &= \left.\frac{\overline{\varphi_2}}{\partial\overline{z}}\right|_{\overline{z}=\overline{z^*}} = 1\\ \left.\frac{\partial\overline{\varphi_1}}{\partial\overline{z}}\right|_{\overline{z}=\overline{z^*}} &= \overline{\varphi_2}|_{\overline{z}=\overline{z^*}} = 0.\end{aligned}$$

In particular, $\overline{f_1}(\overline{z})$ and $\overline{f_2}(\overline{z})$ are linear combinations of $\overline{\varphi_1}(\overline{z})$ and $\overline{\varphi_2}(\overline{z})$.
If $x^* = (x_1^*, x_2^*) \in \Omega$ is a critical point of u, then it holds that

$$\begin{aligned}\overline{f_1}(\overline{z^*}) &= \varphi(z^*, \overline{z^*}) = \left. e^{-u/2}\right|_{x=x^*}\\ \frac{\partial\overline{f_1}}{\partial\overline{z}}(\overline{z^*}) &= \varphi_{\overline{z}}(z^*, \overline{z^*}) = \left.\frac{\partial}{\partial\overline{z}}e^{-u/2}\right|_{x=x^*} = 0\\ \overline{f_2}(\overline{z^*}) &= \varphi_{\overline{z}}(z^*, \overline{z^*}) = \left.\frac{\partial}{\partial\overline{z}}e^{-u/2}\right|_{x=x^*} = 0\\ \frac{\partial\overline{f_1}}{\partial\overline{z}}(\overline{z^*}) &= \varphi_{z\overline{z}}(z^*, \overline{z^*}) = -\frac{1}{4}\Delta e^{-u/2}\Big|_{x=x^*} = -\frac{1}{8}e^{-u/2}\Delta u\Big|_{x=x^*}\\ &= \frac{1}{8}e^{u/2}\Big|_{x=x^*},\end{aligned}$$

and, therefore,

$$\begin{aligned}\overline{f_1}(\overline{z}) &= c\overline{\varphi_1}(\overline{z})\\ \overline{f_2}(\overline{z}) &= \frac{1}{8}c^{-1}\overline{\varphi_2}(\overline{z})\end{aligned}$$

for $c = e^{-u/2}\big|_{x=x^*}$. This relation means

$$f_1 = c\varphi_1$$
$$f_2 = \frac{c^{-1}}{8}\varphi_2$$

and, therefore, we obtain

$$e^{-u/2} = c\,|\varphi_1|^2 + \frac{c^{-1}}{8}\,|\varphi_2|^2 \tag{2.173}$$

by (2.172).

For the proof of Theorem 2.7, let (λ, v) be the solution to (2.155). We apply the above argument replacing u by v and taking x_* as the maximum point of v. Then, we obtain

$$e^{-v/2} = c\,|\varphi_1|^2 + \frac{\sigma c^{-1}}{8}\,|\varphi_2|^2$$

similarly, where

$$\sigma = \frac{\lambda}{\int_\Omega e^v}$$
$$c = e^{-\|v\|_\infty/2}$$

and $\{\varphi_1(z), \varphi_2(z)\}$ is a system of fundamental solutions of (2.170) defined for

$$s(z) = v_{zz} - \frac{1}{2}v_{zz}^2. \tag{2.174}$$

Given the solution sequence $\{(\lambda_k, v_k)\}$, we exclude the boundary blowup points of $\{v_k\}$, using the reflection argument[114] and the L^1 boundedness of the right-hand side.[73] Thanks to the maximum principle and the classical Montel's theorem for holomorphic functions, this boundary estimate implies the compactness of the family of holomorphic functions $\{s_k(z)\}$ defined (2.174) for $v = v_k$ and that of the analytic functions $\{\varphi_{1k}(z), \varphi_{2k}(z)\}$ defined by (2.170)-(2.171). Passing to a subsequence, we obtain

$$\varphi_{1k} \to \varphi_{10}$$
$$\varphi_{2k} \to \varphi_{20}$$

locally uniformly in Ω. Here, we have

$$c_k = e^{-\|v_k\|_\infty} \to 0,$$

with the relation

$$e^{-v_k/2} = c_k\,|\varphi_{1k}|^2 + \frac{\sigma_k c_k^{-1}}{8}\,|\varphi_{2k}|^2.$$

Since $\{v_k\}$ is uniformly bounded near $\partial\Omega$ and zero points of the analytic function φ_{20} are discrete, there arise

$$\sigma_k c_k^{-1} \approx 1$$

and the finiteness of the blowup points of $\{v_k\}$.

Each blowup point is thus isolated, and the classification of the singular limit of the solution (2.157)-(2.158) is obtained by the residue analysis. More precisely, since the limit function $s_0(z)$ of $\{s_k(z)\}$ is holomorphic, the singular point of $s_0(z)$ arising as a blowup point of $\{v_k\}$ is removable. This removable singular point is formally a pole of the second order, and then, (2.157)-(2.158) follow, see.[213] The other proof uses the Pohozaev identity to detect (2.158), see.[191]

Geometric Structure

Writing

$$\psi_1 = c^{1/2} 8^{1/4} \varphi_1$$
$$\psi_2 = c^{-1/2} 8^{-1/4} \varphi_2$$

in (2.168), we have

$$\begin{aligned} W(\psi_1, \psi_2) &= W(\varphi_1, \varphi_2) = 1 \\ \left(\frac{1}{8}\right)^{1/2} e^{u/2} &= \left\{ c\left(\frac{1}{8}\right)^{-1/2} |\varphi_1|^2 + c^{-1}\left(\frac{1}{8}\right)^{1/2} |\varphi_2|^2 \right\}^{-1} \\ &= \frac{1}{|\psi_1|^2 + |\psi_2|^2}, \end{aligned}$$

and, therefore,

$$\frac{|F'|}{1+|F|^2} = \frac{W(\psi_1, \psi_2)}{|\psi_1|^2 + |\psi_2|^2} = \left(\frac{1}{8}\right)^{1/2} e^{u/2} \tag{2.175}$$

for $F = \psi_2/\psi_1$. Thus we can transform (2.155) to

$$\rho(F)|_{\partial\Omega} = \left(\frac{1}{8} \cdot \frac{\lambda}{\int_\Omega e^v}\right)^{1/2} \tag{2.176}$$

using

$$u = v + \log\lambda - \log \int_\Omega e^v$$

with $v|_{\partial\Omega} = 0$, where

$$\rho(F) = \frac{|F'|}{1+|F|^2}.$$

This $\rho(F)$ describes spherical derivative of the meromorphic function $F = F(z)$. More precisely, if $\overline{\mathbf{C}}$ and $d\sigma^2$ denote the Riemann sphere with the south and north poles $(0,0,0)$ and $(0,0,1)$ and its standard metric, respectively, and if $\tau : \overline{\mathbf{C}} \to \mathbf{C} \cup \{\infty\}$ is the stereographic projection, then, the conformal transformation $\overline{F} = \tau^{-1} \circ F$ induces the relation

$$\frac{d\sigma}{ds} = \rho(F), \tag{2.177}$$

where $ds^2 = dx_1^2 + dx_2^2$ stands for the Euclidean metric in $\mathbf{C}$. In particular, $\rho(F)$ is invariant under the $O(3)$ transformation on $\overline{\mathbf{C}}$.
If $\omega \subset\subset \Omega$ is a sub-domain, then the immersed length of $\overline{F}(\partial\omega)$ and the immersed area of $\overline{F}(\omega)$ are defined by

$$\ell_1(\partial\omega) = \int_{\partial\omega} \rho(F)ds$$
$$m_1(\omega) = \int_{\omega} \rho(F)^2 dx,$$

respectively, and, therefore, it follows that

$$\ell_1(\partial\omega)^2 \geq 4m_1(\omega)\,(\pi - m_1(\omega)) \tag{2.178}$$

from the isoperimetric inequality on $\overline{\mathbf{C}}$. Putting

$$\ell(\partial\omega) = \int_{\partial\omega} p^{1/2} ds$$
$$m(\omega) = \int_{\omega} p dx$$

with $p = e^u$, we thus obtain

$$\ell(\partial\omega)^2 \geq \frac{1}{2} m(\omega)\,(8\pi - m(\omega)) \tag{2.179}$$

by (2.175) and (2.178).

Spectral Analysis

The above (2.179) is a special case of Bol's inequality valid on the surface with the Gaussian curvature less than or equal to $1/2$. Taking the non-parametric case, in particular, we obtain (2.179) if $p = p(x) > 0$ is a C^2 function defined on the domain $\Omega \subset \mathbf{R}^2$ of which boundary is composed of a finite number of Jordan curves,

$$-\Delta \log p \leq p \qquad \text{in } \Omega, \tag{2.180}$$

and $\omega \subset\subset \Omega$ is a sub-domain.
Then, Schwarz' symmetrization using the metric $d\sigma = p(x)^{1/2} ds$ guarantees Bandle's isoperimetric inequality.[14]

Theorem 2.15. *If $\Omega \subset \mathbf{R}^2$ is a domain with the boundary $\partial\Omega$ composed of a finite number of Jordan curves and $p = p(x) > 0$ is a C^2 function satisfying (2.180), then it holds that*

$$\lambda \equiv \int_{\Omega} p < 8\pi \quad \Rightarrow \quad \nu_1(p,\Omega) \geq \nu_1(p^*,\Omega^*), \tag{2.181}$$

where

$$\nu_1(p,\Omega) = \inf\left\{ \int_{\Omega} |\nabla v|^2 \mid v \in H_0^1(\Omega),\ \int_{\Omega} v^2 p = 1 \right\}$$

and (p^*, Ω^*) *is determined by*

$$\Omega^* = \left\{ x \in \mathbf{R}^2 \mid |x| < 1 \right\}$$
$$p^* = \frac{\lambda e^{u^*}}{\int_{\Omega^*} e^{u^*}} \tag{2.182}$$

and

$$-\Delta u^* = \frac{\lambda e^{u^*}}{\int_{\Omega^*} e^{u^*}} \quad \text{in } \Omega^*, \qquad u^* = 0 \quad \text{on } \partial\Omega^* \tag{2.183}$$

Here, each $\lambda \in (0, 8\pi)$ admits a unique solution $u^* = u^*_\lambda(x)$ to (2.183), which guarantees the well-definednes of $\nu_1(p^*, \Omega^*)$ in (2.181). This $u^*_\lambda(x)$ is radially symmetric, described explicitly, and satisfies

$$\lim_{\lambda \downarrow 0} u^*(x) = 0 \qquad \text{uniformly in } x \in \overline{\Omega^*}$$
$$\lim_{\lambda \uparrow 8\pi} u^*(x) = 4 \log \frac{1}{|x|} \quad \text{locally uniformly in } x \in \overline{\Omega^*} \setminus \{0\}.$$

Furthermore, the value $\nu_1(p^*, \Omega^*)$ is associated with the Laplace-Beltrami operator on $\overline{\mathbf{C}}$, and, therefore, using the separation of variables and the stereographic projection, we obtain

$$0 < \lambda < 4\pi \;\Rightarrow\; \nu_1(p^*, \Omega^*) > 1. \tag{2.184}$$

More precisely, $\nu_1(p^*, \Omega^*)$ is realized as the first eigenvalue of the eigenvalue problem

$$-\Delta\varphi = \nu\zeta_0\varphi \quad \text{in } \Omega^*, \qquad \varphi = 0 \quad \text{on } \partial\Omega^*, \tag{2.185}$$

where

$$\zeta_0 = \zeta_0(x) = \frac{8\rho}{\left(|x|^2 + \rho\right)^2}$$

and $\rho > 0$ is a constant determined by

$$\lambda = \int_{\Omega^*} \zeta_0.$$

Then, putting

$$\varphi(x) = \Phi(\xi) e^{\imath m\theta}$$
$$x = re^{\imath\theta}, \quad \xi = \frac{\rho - r^2}{\rho + r^2}$$
$$\Lambda = 1/\nu, \tag{2.186}$$

we obtain the associated Legendre equation

$$\left[(1-\xi^2)\Phi_\xi\right]_\xi + \left[2/\Lambda - m^2/(1-\xi^2)\right]\Phi = 0, \quad \xi_\rho < \xi < 1$$
$$\Phi(1) = 1$$
$$\Phi(\xi_\rho) = 0 \tag{2.187}$$

for $\xi_\rho = (\rho-1)/(\rho+1)$, see.[14] This formulation is natural because the Laplace-Beltrami operator on two-dimensional round sphere is immersed into the three dimensional Laplacian in the Cartesian coordinate, which is involved by the associated Legendre equation through the separation of variables using polar coordinates.[303] Thus if $\Phi = \Phi(\xi)$ denotes a solution to the first equation of (2.187) for $\Lambda = 1$, $m = 0$, and $\Phi(1) = 1$, then $\nu_1(p^*, \Omega^*) > 1$ is equivalent to

$$\Phi(\xi) > 0, \qquad \xi_\rho < \xi < 1.$$

Since such Φ is given by $P_0(\xi) = \xi$, this relation means $\xi_\rho > 0$, and, therefore, (2.184) follows from

$$\lambda < 4\pi \quad \Leftrightarrow \quad \rho > 1 \quad \Leftrightarrow \quad \xi_\rho > 0.$$

Theorem 2.8, on the other hand, is proven by the bifurcation analysis and an a priori estimate. More precisely, we show that the linearized operator is non-degenerate if $0 < \lambda < 8\pi$ because the solution set is compact in this range by Theorem 2.7. Actually, the linearized operator concerning (2.155) around the solution $v = \overline{v}$ is defined by

$$\mathscr{L}\psi = -\Delta\psi - \lambda\left(\frac{e^{\overline{v}}\psi}{\int_\Omega e^{\overline{v}}} - \frac{\int_\Omega e^{\overline{v}}\psi}{(\int_\Omega e^{\overline{v}})^2}\cdot e^{\overline{v}}\right)$$

with the domain $\mathscr{L} = H^2(\Omega)\cap H_0^1(\Omega)$ in $L^2(\Omega)$, and the transformation

$$\varphi = \psi - \frac{1}{|\Omega|}\int_\Omega \psi$$

used in §1.2.8 induces the eigenvalue problem (1.174). Using $p = p(x)$ defined by

$$p = \overline{u} = \frac{\lambda e^{\overline{v}}}{\int_\Omega e^{\overline{v}}},$$

see (1.174), this problem is written as

$$-\Delta\varphi = \mu p\varphi \ \text{ in } \Omega, \quad \varphi = \text{constant on } \partial\Omega$$
$$\int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu} = 0, \tag{2.188}$$

and this $p = p(x) > 0$ is provided with the property (2.180). Thus we have only to show that $\mu = 1$ is not an eigenvalue of (2.188) provided that $0 < \lambda = \int_\Omega p < 8\pi$.
Since the first eigenvalue of this eigenvalue problem is $\mu = 0$, we show that the second eigenvalue is greater than 1 in this case.[301] This property means

$$0 < \lambda < 8\pi \ \Rightarrow \ \mu_2(p,\Omega) > 1, \tag{2.189}$$

where

$$\mu_2(p,\Omega) = \inf\left\{\int_\Omega |\nabla v|^2 \mid v \in H_c^1(\Omega),\ \int_\Omega vp = 0,\ \int_\Omega v^2 p = 1\right\}$$

for

$$H_c^1(\Omega) = \left\{v \in H^1(\Omega) \mid v = \text{constant on } \partial\Omega\right\},$$

which comprises of the key lemma for the proof of Theorem 2.8.
For simplicity, we assume that $\Omega \subset \mathbf{R}^2$ is simply-connected. To derive (2.189), first, we show that the associated eigenfunction to $\mu_2(p,\Omega)$ has exactly two nodal domains. Next, if any of them is not enclosed inside Ω, then we can apply (2.181) with (2.184) to either and obtain (2.189). The same argument is valid if one of them, denoted by ω, is enclosed inside Ω and it holds that

$$\int_\omega p < 4\pi.$$

For the rest case of

$$\int_\omega p \geq 4\pi,$$

we take the doubly-connected domain $\hat{\omega} = \Omega \setminus \omega$, and obtain the eigenvalue problem

$$-\Delta\varphi = \mu p(x)\varphi, \quad \varphi > 0 \qquad \text{in } \hat{\omega}$$
$$\varphi = 0 \quad \text{on } \partial\hat{\omega} \setminus \partial\Omega, \qquad \varphi = \text{constant} \quad \text{on } \partial\Omega.$$

Thus $\mu = \mu_2(p,\Omega)$ is the first eigenvalue of this eigenvalue problem, with $p = p(x) > 0$ satisfying (2.180) and

$$\int_{\hat{\omega}} p < 4\pi. \tag{2.190}$$

Here, we modify the Schwarz symmetrization on $\overline{\mathbf{C}}$, using the above described nodal domain property. In this case, (2.189) is reduced to examine an eigenvalue problem of the Laplace-Beltrami operator provided with a modified Dirichlet boundary condition defined on an annular domain of $\overline{\mathbf{C}}$, which is contained in the chemi-sphere by (2.190). Through the transformation (2.186), this problem is realized as the associated Legendre equation, and, then, (2.189) is shown.[301,303] This proof of (2.189) is valid even to the multiply-connected $\Omega \subset \mathbf{R}^2$, and also to the case of $\lambda = 8\pi$, (if the solution exists), see[47,308] for details.

2.3.5. Alexandroff-Bol's Inequality

Using Bol's inequality and its refinement of Alexandrov's inequality, we obtain the (bubbled) Harnack principle[303] and the $\sup + \inf$ inequality.[283] For the moment, we specify the volume and the line elements by dx and ds, respectively.

Bol's Inequality

First, we recall the analytic form of Bol's inequality described in §2.3.4.

Theorem 2.16. *If $\Omega \subset \mathbf{R}^2$ is a simply-connected domain, $p = p(x) > 0$ is a C^2 function satisfying*

$$-\Delta \log p \leq p \qquad \text{in } \Omega, \tag{2.191}$$

and $\omega \subset\subset \Omega$ is a sub-domain, then it holds that

$$\ell(\partial\omega)^2 \geq \frac{1}{2} m(\omega)\left(8\pi - m(\omega)\right), \tag{2.192}$$

where

$$\ell(\partial\omega) = \int_{\partial\omega} p^{1/2} ds$$
$$m(\omega) = \int_{\omega} p dx.$$

Inequality (2.192) holds also if Ω is multiply-connected, by decomposing it into simply-connected domains.[14] Nehari's isoperimetric inequality[218] described below is the limiting case of (2.191)-(2.192) to the zero Gaussian curvature.

Theorem 2.17. *If $\Omega \subset \mathbf{R}^2$ is a simply-connected domain, $h = h(x)$ is harmonic in Ω, and $\omega \subset \Omega$ is a su-domain, then it holds that*

$$\left\{\int_{\partial\omega} e^{h/2} ds\right\}^2 \geq 4\pi \int_{\omega} e^h dx. \tag{2.193}$$

For the proof of the aobve theorem, we define the univalent function $g(z)$ in Ω by $|g'|^2 = e^h$. Then,

$$\int_{\partial\omega} e^{h/2} ds = \int_{\partial\omega} |g'|\, ds$$
$$\int_{\omega} e^h dx = \int_{\omega} |g'|^2\, dx$$

indicate the immersed length of $g(\partial\omega)$ and the immersed area of $g(\omega)$, and (2.193) follows from the isoperimetric inequality on the plane. We now describe the analytic proof[13] of Theorem 2.16.

Proof of (2.193)⇒(2.192): We may suppose that $\partial\omega$ is C^1. Define $h = h(x)$ by

$$\Delta h = 0 \quad \text{in } \omega, \qquad h = \log p \quad \text{on } \partial\omega$$

and put $q = pe^{-h}$. This formulation implies

$$-\Delta \log q \leq qe^h \quad \text{in } \omega, \qquad q = 1 \quad \text{on } \partial\omega \tag{2.194}$$

by (2.191).
Using the right-continuous non-increasing functions

$$K(t) = \int_{\{q>t\}} qe^h dx$$
$$\mu(t) = \int_{\{q>t\}} e^h dx$$

defined by $\{q > t\} = \{x \in \omega \mid q(x) > t\}$, we obtain

$$\begin{aligned} -K'(t) &= \int_{\{q=t\}} \frac{qe^h}{|\nabla q|} ds \\ &= t \int_{\{q=t\}} \frac{e^h}{|\nabla q|} ds = -t\mu'(t) \end{aligned} \tag{2.195}$$

a.e. t by the co-area formula.[87] We have, on the other hand,

$$\int_{\{q>t\}} (-\Delta \log q)dx = \int_{\{q=t\}} \frac{|\nabla q|}{q} ds = \frac{1}{t} \int_{\{q=t\}} |\nabla q|\, ds$$

a.e. $t > 1$, using Green's formula and Sard's lemma. Thus we obtain

$$\frac{1}{t} \int_{\{q=t\}} |\nabla q|\, ds \leq \int_{\{q>t\}} q e^h dx = K(t)$$

by (2.194), and, therefore,

$$\begin{aligned} -K'(t)K(t) &\geq \frac{1}{t} \int_{\{q=t\}} |\nabla q|\, ds \cdot t \int_{\{q=t\}} \frac{e^h}{|\nabla q|} ds \\ &\geq \left\{ \int_{\{q=t\}} e^{h/2} ds \right\}^2 \\ &\geq 4\pi \int_{\{q>t\}} e^h dx = 4\pi\mu(t) \end{aligned} \tag{2.196}$$

a.e. $t > 1$ by Schwarz' and Nehari's inequalities. Thus from (2.195)-(2.196) it follows that

$$\frac{d}{dt} \left\{ \mu(t)t - K(t) + \frac{K(t)^2}{8\pi} \right\} = \mu(t) + \frac{1}{4\pi} K(t)K'(t) \leq 0 \tag{2.197}$$

for a.e. $t > 1$.

We obtain, on the other hand,

$$K(t+0) = K(t) \leq K(t-0),$$

while

$$j(t) = K(t) - \mu(t)t = \int_{\{q>t\}} (q-t)e^h dx$$

is continuous: $j(t+0) = j(t) = j(t-0)$. Thus (2.197) guarantees

$$\left[-j(t) + \frac{K(t)^2}{8\pi} \right]_{t=1}^{\infty} = j(1) - \frac{K(1)^2}{8\pi} \leq 0. \tag{2.198}$$

Using

$$\begin{aligned} &j(1) = \int_{\{q>1\}} (q-1)e^h dx \geq \int_\omega (q-1)e^h dx = m(\omega) - \int_\omega e^h dx \\ &K(1)^2 \leq m(\omega)^2, \end{aligned}$$

we deduce

$$\begin{aligned} m(\omega) - \frac{m(\omega)^2}{8\pi} &\leq \int_\omega e^h dx \\ &\leq \frac{1}{4\pi} \left\{ \int_{\partial\omega} e^{h/2} ds \right\}^2 = \frac{\ell(\partial\omega)^2}{4\pi}. \end{aligned}$$

This inequality means (2.192). □

Similarly to (2.198), we obtain

$$j(t) \le \frac{K(t)^2}{8\pi} \tag{2.199}$$

for any $t > 1$. This inequality implies the mean value theorem.[303] We note that the converse is also true in the following theorem.

Theorem 2.18. *If* $\Omega \subset \mathbf{R}^2$ *is an open set,* $p = p(x) > 0$ *is a* C^2 *function satisfying (2.191), and* $B = B(x_0, r) \subset\subset \Omega$*, then it holds that*

$$\log p(x_0) \le \frac{1}{|\partial B|}\int_{\partial B} \log p ds - 2\log\left(1 - \frac{1}{8\pi}\int_B p dx\right)_+. \tag{2.200}$$

Proof: Using (2.199) for $\omega = B$, we obtain

$$\mu(t) \ge \frac{K(t)^2}{t}\left(\frac{1}{K(t)} - \frac{1}{8\pi}\right) \tag{2.201}$$

for any $t > 1$. Furthermore,

$$\begin{aligned} J(t) &= \frac{\mu}{K(t)} - \frac{\mu(t)}{8\pi} \\ &= \frac{1}{t} - \frac{j(t)}{tK(t)} - \frac{\mu(t)}{8\pi} \end{aligned}$$

is right-continuous: $J(t+0) = J(t)$, and it holds also that

$$\begin{aligned} J(t-0) - J(t) &= \frac{j(t)}{t}\left\{\frac{1}{K(t)} - \frac{1}{K(t-0)}\right\} + \frac{1}{8\pi}(\mu(t) - \mu(t-0)) \\ &= \frac{j(t)}{t}\left\{\frac{1}{j(t) + \mu(t)t} - \frac{1}{j(t) + \mu(t-0)t}\right\} \\ &\quad + \frac{1}{8\pi}(\mu(t) - \mu(t-0)) \\ &= -(\mu(t) - \mu(t-0)) \cdot \left\{\frac{j(t)}{K(t)K(t-0)} - \frac{1}{8\pi}\right\}. \end{aligned}$$

Using

$$\begin{aligned} K(t-0) - K(t) &= \int_{\{q=t\}} qe^h dx \\ &= t(\mu(t-0) - \mu(t)) \ge 0 \end{aligned}$$

and $j(t) \ge 0$, we obtain

$$J(t-0) - J(t) \le -(\mu(t-0) - \mu(t)) \cdot \left(\frac{1}{8\pi} - \frac{j(t)}{K(t)^2}\right) \le 0$$

by (2.199). On the other hand, we have

$$\begin{aligned} J'(t) &= \mu'(t)\left(\frac{1}{K(t)} - \frac{1}{8\pi}\right) - \mu(t)\frac{K'(t)}{K(t)^2} \\ &\geq \mu'(t)\cdot\frac{t\mu(t)}{K(t)^2} - \mu(t)\cdot\frac{K'(t)}{K(t)^2} = 0 \end{aligned}$$

for a.e. $t > 1$ by (2.199) and (2.195).
These relations imply

$$\begin{aligned} \lim_{t\uparrow t_0} J(t) &= \lim_{t\uparrow t_0}\frac{\mu'(t)}{K'(t)} = \frac{1}{t_0} \\ &\geq J(t) = \mu(t)\left(\frac{1}{K(t)} - \frac{1}{8\pi}\right) \end{aligned}$$

for $1 \leq t \leq t_0$ by (2.195), where $t_0 = \max_{\overline{B}} q$. Here, the right-hand side is estimated from below by

$$\frac{K(t)^2}{t}\left(\frac{1}{K(t)} - \frac{1}{8\pi}\right)_+^2$$

using (2.201). Then, putting $t = 1$, we obtain

$$\frac{1}{t_0} \geq \left(1 - \frac{K(1)}{8\pi}\right)_+^2 \geq \left(1 - \frac{m}{8\pi}\right)_+^2,$$

where

$$m = \int_B p dx.$$

This inequality means

$$\left(1 - \frac{m}{8\pi}\right)_+^2 \geq t_0 = \max_{\overline{B}} pe^{-h} \geq p(x_0)e^{-h(x_0)}$$

or, equivalently,

$$\log p(x_0) \leq h(x_0) - 2\log\left(1 - \frac{1}{8\pi}\int_B p dx\right)_+.$$

Here, using the mean value theorem to the harmonic function, we obtain

$$\begin{aligned} h(x_0) &= \frac{1}{|\partial B|}\int_{\partial B} h ds \\ &= \frac{1}{|\partial B|}\int_{\partial B} \log p ds, \end{aligned}$$

and the proof of (2.200) is complete. □

Similarly to the harmonic case, the mean value theorem (2.200) implies the following Harnack inequality.

Theorem 2.19. *If $v = v(x) \in C^2(B) \cap C(\overline{B})$ satisfies*

$$0 \leq -\Delta v \leq \frac{\lambda e^v}{\int_\Omega e^v}, \ \ v \geq 0 \qquad \textit{in } B$$

for $B = B(0,R) \subset \mathbf{R}^2$, then it holds that

$$v(0) \leq \frac{R+|x|}{R-|x|} v(x) - 2\log\left(1 - \frac{\lambda}{8\pi}\right)_+ \tag{2.202}$$

for $x \in B$.

Proof: We obtain

$$-\Delta \log p \leq p \qquad \text{in } B$$

for $p = \lambda e^v / \int_\Omega e^v$, and, therefore, (2.200) is applicable. It holds that

$$\log p(0) \leq \frac{1}{|\partial B|} \int_{\partial B} \log p ds - 2\log\left(1 - \frac{\lambda}{8\pi}\right)_+,$$

or, equivalently,

$$v(0) \leq \frac{1}{|\partial B|} \int_{\partial B} v ds - 2\log\left(1 - \frac{\lambda}{8\pi}\right)_+. \tag{2.203}$$

Since $v(x) \geq 0$ is super-harmonic, on the other hand, it holds that

$$\begin{aligned} v(re^{\imath\theta}) &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} v(Re^{\imath\varphi}) d\varphi \\ &\geq \frac{R-r}{R+r} \frac{1}{|\partial B|} \int_{\partial B} v ds \end{aligned}$$

for $0 \leq r < R$. This formula implies (2.202). □

From the standard argument, Harnack's inequality (2.202) implies the Harnack principle stated as follows.

Theorem 2.20. *If $\Omega \subset \mathbf{R}^2$ is an open set and $v_k = v_k(x)$, $k = 1, 2, \ldots$, are C^2 functions satisfying*

$$0 \leq -\Delta v_k \leq \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k}}, \quad v_k \geq 0 \qquad \textit{in } \Omega$$

for $\lambda_k > 0$, then passing to a sub-sequence, we obtain the following alternatives, where

$$\mathscr{S} = \{x_0 \in \Omega \mid \textit{there exists } x_k \to x_0 \textit{ such that } v_k(x_k) \to +\infty\}.$$

(i) *$\{v_k\}_k$ is locally uniformly bounded in Ω.*
(ii) *$v_k \to +\infty$ locally uniformly in Ω.*
(iii) *$\mathscr{S} \neq \emptyset$ and $\sharp\mathscr{S} \leq \liminf_k [\lambda_k/(8\pi)]$.*

Alexandrov's Inequality

Alexandrov's inequality, see[14] for the proof, is a sharp form of Bol's inequality. First, we introduce the metric $d\sigma^2 = p(x)ds^2$ on $B = B(0,1) \subset \mathbf{R}^2$, using $p = p(x) > 0$ such that $C^2(B) \cap C(\overline{B})$. This formulation implies that its Gaussian curvature, total volume m, the boundary length ℓ are defined by

$$K = -\frac{\Delta \log p}{2p}$$
$$m = \int_B p dx$$
$$\ell = \int_{\partial B} p^{1/2} ds.$$

Then, if

$$\alpha = 2\pi - m_\mu^+(B) > 0$$

for

$$m_\mu^+(B) = \int_{\{K>\mu\}} (K(x) - \mu)\, p dx,$$

it holds that

$$\ell^2 \geq (2\alpha - \mu m)m, \tag{2.204}$$

where $\mu \in \mathbf{R}$.
In the case of $K \leq 1/2$, it follows that

$$m_\mu^+(B) = \left(\frac{1}{2} - \mu\right) \int_B p dx = \left(\frac{1}{2} - \mu\right) m$$
$$\alpha = 2\pi + \left(\mu - \frac{1}{2}\right) m$$

for $\mu < 1/2$. Then, (2.204) reads;

$$\ell^2 \geq m \cdot (4\pi + (\mu - 1)m)\,.$$

Letting $\mu \uparrow 1/2$, we obtain

$$\ell^2 \geq \frac{1}{2} m(8\pi - m).$$

This formula means Bol's inequality on B.
In the actual application,[14] we take μ_* satisfying

$$\int_{\{K>\mu_*\}} p dx \leq m/2$$
$$\int_{\{K<\mu_*\}} p dx \leq m/2.$$

In this case, it is obvious that

$$\{K > \mu_*\} \neq B$$
$$\{K < \mu_*\} \neq B,$$

and, therefore,

$$\{K \le \mu_*\} \neq \emptyset$$
$$\{K \ge \mu_*\} \neq \emptyset.$$

This relation implies

$$\frac{a}{2} \le \mu_* \le \frac{b}{2},$$

provided that

$$\frac{a}{2} \le K(x) \le \frac{b}{2},$$

where $a, b > 0$ are constants.
There are $C, D \ge 0$ such that

$$\int_{\{K>\mu_*\}} p dx = \frac{m}{2} - C$$
$$\int_{\{K<\mu_*\}} p dx = \frac{m}{2} - D$$

and, then, it holds that

$$\begin{aligned} 2\alpha - \mu_* m &= 4\pi - 2m^+_{\mu_*}(B) - \mu_* m \\ &= 4\pi - 2\int_{\{K>\mu_*\}} (K-\mu_*) p dx - \mu_* m \\ &= 4\pi - 2\int_{\{K\ge\mu_*\}} (K-\mu_*) p dx - \mu_* m \\ &= 4\pi - 2\int_{\{K\ge\mu_*\}} K p dx + 2\mu_* D. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \int_{\{K<\mu_*\}} K p dx &\ge \frac{a}{2}\int_{\{K<\mu_*\}} p dx = \frac{a}{2}\left(\frac{m}{2} - D\right) \\ &\ge \frac{a}{2b}\int_B K p dx - \frac{a}{2} D, \end{aligned}$$

and, therefore,

$$\int_{\{K\ge\mu_*\}} K p dx \le \left(1 - \frac{a}{2b}\right)\int_B K p dx + \frac{a}{2} D.$$

Thus it follows that

$$\begin{aligned} 2\alpha - \mu_* m &\ge 4\pi - 2\left(1 - \frac{a}{2b}\right)\int_B K p dx + (2\mu_* - a) D \\ &\ge 4\pi - 2\left(1 - \frac{a}{2b}\right)\int_B K p dx, \end{aligned}$$

and, consequently,

$$\int_B Kpdx \le \alpha_0/2 \;\Rightarrow\; 2\alpha - \mu_* m \ge \gamma_0, \tag{2.205}$$

where

$$4\pi < \alpha_0 < \frac{4\pi}{1-\frac{a}{2b}}$$
$$\gamma_0 = 4\pi - \left(1 - \frac{a}{2b}\right)\alpha_0 > 0.$$

The inclusion (2.205) implies the following lemma.

Lemma 2.1. *Let $B = B(0,1) \subset \mathbf{R}^2$ and $a, b > 0$ be constants. Then, there is $C_0 > 0$ such that*

$$-\Delta v = V(x)e^v, \;\; a \le V(x) \le b \qquad \text{in } B$$
$$\int_B V(x)e^v \le \alpha_0$$

implies

$$v(0) \le C_0, \tag{2.206}$$

where $v \in C^2(B) \cap C(\overline{B})$.

Proof: We can apply (2.205) to

$$p = e^u$$
$$K = V/2.$$

In fact, the function

$$f(r) = 4\pi - 2\int_{\{K>\mu_*\}} (K-\mu_*)pdx - \mu_* \int_\Omega pdx$$

is strictly decreasing in $r \in [0,1)$, and, therefore,

$$f(r) \ge f(1) = 2\alpha - \mu_* m \ge \gamma_0.$$

Putting

$$A(r) = \int_0^r \int_{\partial B(0,r')} pdsdr' = \int_{B(0,r)} pdx,$$

on the other hand, we obtain

$$A'(r) = \int_{\partial B(0,r)} pds \ge \frac{1}{2\pi r}\left\{\int_{\partial B(0,r)} p^{1/2}ds\right\}^2 \ge \frac{1}{2\pi r} f(r)A(r) \tag{2.207}$$

for $0 \le r < 1$ by (2.204). This inequality implies

$$\begin{aligned} -\frac{1}{2\pi}\log r_0 &\le \int_{r_0}^1 \frac{A'(r)}{f(r)A(r)}dr \\ &= \int_{r_0}^1 \frac{1}{f(r)}\{\log A(r)\}'\,dr \\ &= \frac{\log A(1)}{f(1)} - \frac{\log A(r_0)}{f(r_0)} + \int_{r_0}^1 \frac{\log A(r)}{f(r)^2} f'(r)dr \end{aligned}$$

for $r_0 \in (0,1)$. Since

$$\begin{aligned} A(r_0) &\sim \pi r_0^2 p(0) \\ f(r_0) &\sim 4\pi - O\left(r_0^2\right) \end{aligned}$$

as $r_0 \downarrow 0$, it holds that

$$\begin{aligned} \lim_{r_0\downarrow 0}\left\{\frac{\log A(r_0)}{f(r_0)} - \frac{1}{2\pi}\log r_0\right\} &= \lim_{r_0\downarrow 0}\frac{1}{4\pi}\log\frac{A(r_0)}{r_0^2} \\ &= \frac{1}{4\pi}\left(\log p(0) + \log \pi\right), \end{aligned}$$

and, therefore,

$$u(0) = \log p(0) \le \frac{4\pi}{f(1)}\log\left(\frac{A(1)}{\pi}\right) + 4\pi\int_0^1 \frac{\log A(r)}{f(r)^2} f'(r)dr.$$

We have, on the other hand,

$$\begin{aligned} A(r) \le A(1) &= \int_\Omega p dx \le \frac{1}{2a}\int_B K p dx \le \frac{\alpha_0}{a} \\ \frac{\log A(r)}{f(r)^2} f'(r) &= \frac{\log\frac{\alpha_0}{A(r)} + \log\frac{1}{\alpha_0}}{f(r)^2}\left(-f'(r)\right) \\ &\le \gamma_0^{-2}\log\frac{\alpha_0}{A(r)}\cdot\left(-f'(r)\right), \end{aligned}$$

and

$$\begin{aligned} 0 &\le -f'(r) \\ &= 2\int_{\{K>\mu_*\}\cap\partial B_r}(K-\mu_*)pds + \mu_*\int_{\partial B_r} pds \\ &\le 2\cdot\int_{\{K>\mu_*\}\cap\partial B_r}\left(\frac{a}{2}-\frac{b}{2}\right)pds + \frac{b}{2}\int_{\partial B_r} pds \\ &\le \left(\frac{3}{2}b - a\right)\int_{\partial B_r} pds \\ &= \left(\frac{3}{2}b - a\right)A'(r). \end{aligned}$$

Using the transformation $t = \frac{A(r)}{\alpha_0}$, we thus obtain

$$\int_0^1 \frac{\log A(r)}{f(r)^2}\cdot f'(r)dr \le \gamma_0^{-2}\left(\frac{3}{2}b - a\right)\cdot\frac{1}{\alpha_0}\int_0^1 \log\frac{1}{t}dt < +\infty,$$

and, therefore, (2.206). □

If $V(x)$ is restricted to a compact family of $C(\overline{B})$, then the blowup analysis based on the classification of the entire solution, see §2.3.7, provides with an alternative proof of the above lemma. The conclusion $\alpha_0 > 4\pi$, on the other hand, is regarded as an improvement of Brezis-Merle's rough estimate described in the next paragraph under the cost of $V(x) \geq a > 0$.

Sup + Inf Inequality

Lemma 2.1 implies the sup + inf inequality,[283] see[31,53] for the other version.

Theorem 2.21. *If $\Omega \subset \mathbf{R}^2$ is a bounded domain, $K \subset \Omega$ is a compact set, and $a, b > 0$ are constants, then there are $c_1 = c_1(a,b) \geq 1$ and $c_2 = c_2(a, b, dist(K, \partial\Omega)) > 0$ such that*

$$-\Delta v = V(x)e^v, \ \ a \leq V(x) \leq b \qquad in\ \Omega \tag{2.208}$$

implies

$$\sup_K v + c_1 \inf_\Omega v \leq c_2. \tag{2.209}$$

Proof: From the standard covering argument, the assertion is reduced to the case $\Omega = B(x_0, r)$, $K = \{x_0\}$, $c_2 = c_2(r)$. Assuming $x_0 = 0$, we take

$$\tilde{v}(x) = v(rx) + 2\log r,$$

which satisfies (2.208) for $\Omega = B(0,1)$ and $\tilde{V}(x) = V(rx)$. Thus we have only to consider the case $\Omega = B = B(0,1)$ and $K = \{0\}$. Actually, we show, more strongly,

$$v(0) + \frac{1}{2\pi}\int_{\partial B} v ds \leq c_2 \tag{2.210}$$

in this case.
For $\alpha_0 > 4\pi$ in Lemma 2.1, we define $c_1 \geq 1$ by

$$4\pi(c_1 + 1)/c_1 = \alpha_0.$$

In the case of

$$\int_B Ve^v > 4\pi(c_1 + 1)/c_1, \tag{2.211}$$

we define $r_0 \in (0,1)$ by

$$\int_{B(0,r_0)} Ve^v = 4\pi(c_1 + 1)/c_1$$

and $r_0 = 1$ otherwise. In any case, we obtain

$$\int_{B(0,r_0)} Ve^v \leq 4\pi(c_1 + 1)/c_1.$$

Putting $\tilde{u}(x) = u(r_0 x) + 2\log r_0$ and $\tilde{V}(x) = V(r_0 x)$, we obtain (2.208) for $\Omega = B = B(0,1)$ and also

$$\int_B \tilde{V} e^{\tilde{v}} \leq 4\pi(c_1+1)/c_1 = \alpha_0.$$

This inequality implies

$$\tilde{v}(0) = v(0) + 2\log r_0 \leq c_0 \tag{2.212}$$

by Lemma 2.1.
Now, we define

$$\begin{aligned} G(r) &= v(0) + \frac{c_1}{2\pi r}\int_{\partial B(0,r)} v ds + 2(c_1+1)\log r \\ &= u(0) + c_1 \int_0^{2\pi} v(r,\theta) d\theta + 2(c_1+1)\log r. \end{aligned}$$

Since v is super-harmonic, we obtain

$$\frac{1}{2\pi r}\int_{\partial B(0,r)} v ds \leq v(0)$$

and, therefore,

$$G(r_0) \leq (c_1+1)v(0) + 2(c_1+1)\log r_0 \leq (c_1+1)c_0 \tag{2.213}$$

by (2.212). Thus it holds that (2.210) if $r_0 = 1$, by

$$G(1) = v(0) + \frac{c_1}{2\pi}\int_{\partial B} v ds.$$

In the other case of $r_0 < 1$, we obtain (2.211), and, therefore, it holds that

$$\int_{B(0,r)} V e^v \geq 4\pi(c_1+1)/c_1, \qquad r_0 \leq r \leq 1. \tag{2.214}$$

Using

$$\begin{aligned} G'(r) &= c_1 \int_0^{2\pi} v_r(r,\theta) d\theta + \frac{2(c_1+1)}{r} \\ &= \frac{1}{r}\left\{ \frac{c_1}{2\pi}\int_{\partial B(0,r)} v_r ds + 2(c_1+1) \right\} \end{aligned}$$

and

$$\int_{\partial B(0,r)} v_r ds = \int_{B(0,r)} \Delta v = -\int_{B(0,r)} V e^v,$$

we obtain

$$G'(r) \leq 0, \qquad r_0 \leq r \leq 1$$

by (2.214). This inequality implies

$$u(0) + \frac{1}{2\pi}\int_{\partial B} v ds = G(1) \leq G(r_0) \leq (c_1+1)c_0$$

by (2.213), and the proof is complete. □

2.3.6. Pre-Scaled Analysis

This paragraph is devoted to Brezis-Merle's theorem[32] concerning

$$\begin{aligned} &-\Delta v = V(x)e^v, \quad 0 \le V(x) \le c_0 \qquad \text{in } \Omega \\ &\int_\Omega e^v \le c_1, \end{aligned} \tag{2.215}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain.
First, we show Brezis-Merle's inequality.

Theorem 2.22. *If $\Omega \subset \mathbf{R}^2$ is a bounded domain, $f \in L^1(\Omega)$, and*

$$-\Delta v = f(x) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then it holds that

$$\int_\Omega \exp\left(\frac{4\pi-\delta}{\|f\|_1}|v(x)|\right)dx \le \frac{4\pi^2}{\delta}(\text{diam } \Omega)^2.$$

Proof: We take $B = B(x_0, R)$ containing Ω for $R = \frac{1}{2}\text{diam } \Omega$, 0-extension of $f(x)$ outside Ω, and

$$\overline{v}(x) = \frac{1}{2\pi}\int_B \log\frac{2R}{|x-x'|}\cdot\left|f(x')\right|dx'.$$

Then, it holds that

$$-\Delta\overline{v} = |f| \qquad \text{in } \mathbf{R}^2$$

and $\overline{v} \ge 0$ in B by $2R/|x-x'| \ge 1$ for $x, x' \in B$, and, therefore,

$$|v| \le \overline{v}$$

from the maximum principle. This inequality implies

$$\int_\Omega \exp\left(\frac{4\pi-\delta}{\|f\|_1}|v(x)|\right)dx \le \int_\Omega \exp\left(\frac{4\pi-\delta}{\|f\|_1}\overline{v}(x)\right)dx. \tag{2.216}$$

Next, applying Jensen's inequality, we obtain

$$\begin{aligned} \exp\left(\frac{4\pi-\delta}{\|f\|_1}\overline{v}(x)\right) &= \exp\left(\int_B \frac{4\pi-\delta}{2\pi}\cdot\log\frac{2R}{|x-x'|}\cdot\frac{|f(x')|}{\|f\|_1}dx'\right) \\ &\le \int_B \left(\frac{2R}{|x-x'|}\right)^{2-\frac{\delta}{2\pi}}\frac{|f(x')|}{\|f\|_1}dx', \end{aligned}$$

and the right-hand side of (2.216) is estimated from above by

$$\int_B \frac{|f(x')|}{\|f\|_1}dx'\cdot\int_B\left(\frac{2R}{|x-x'|}\right)^{2-\frac{\delta}{2\pi}}dx.$$

Since $B = B(x_0, R) \subset B(x', 2R)$ for any $x' \in B$, it holds that

$$\begin{aligned}
&\int_\Omega \exp\left(\frac{4\pi-\delta}{\|f\|_1}|v(x)|\right)dx \\
&\leq \int_B \frac{|f(x')|}{\|f\|_1}dx' \cdot \int_{B(x',2R)} \left(\frac{2R}{|x-x'|}\right)^{2-\frac{\delta}{2\pi}} dx \\
&= \frac{4\pi^2}{\delta}(2R)^2 = \frac{4\pi^2}{\delta}(\mathrm{diam}\,\Omega)^2,
\end{aligned}$$

and the proof is complete. □

We show the following theorem as an application.

Theorem 2.23. *If*

$$v^- \in L^1_{loc}(\mathbf{R}^2), \quad V \in L^p(\mathbf{R}^2), \quad e^v \in L^{p'}(\Omega)$$

and

$$-\Delta v = V(x)e^v \qquad in\ \mathbf{R}^2,$$

then it follows that $v^+ \in L^\infty(\mathbf{R}^2)$*, where* $1 < p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$*, and* $v^\pm = \max(\pm v, 0)$.

Proof: Given $\varepsilon \in (0, 1/p')$, we divide $Ve^v \in L^1(\mathbf{R}^2)$ by

$$\begin{aligned}
&Ve^v = f_1 + f_2 \\
&\|f_1\|_{L^1(\mathbf{R}^2)} < \varepsilon \\
&f_2 \in L^\infty(\mathbf{R}^2).
\end{aligned}$$

Fix $x_0 \in \mathbf{R}^2$, and put $B_r = B(x_0, r)$ for simplicity. We apply Theorem 2.22 to $\delta = 4\pi - 1$ and

$$-\Delta v_i = f_i \quad \text{in } B_1, \qquad v_i = 0 \quad \text{on } \partial B_1,$$

and then obtain

$$\int_{B_1} \exp(|v_1|/\varepsilon) \leq C,$$

and in particular,

$$\|v_1\|_{L^1(B_1)} \leq C.$$

Here and henceforth, $C > 0$ denotes a constant independent of x_0, possibly changing from line to line.

We have, on the other hand,

$$\|v_2\|_{L^\infty(B_1)} \leq C$$

from the elliptic regularity, and furthermore, $v_3 = v - v_1 - v_2$ is a harmonic function in B_1. Using the mean value theorem, therefore, we obtain

$$\left\|v_3^+\right\|_{L^\infty(B_{1/2})} \leq C\left\|v_3^+\right\|_{L^1(B_1)}.$$

Now, we use

$$v_3^+ \leq v^+ + |v_1| + |v_2|$$

and

$$p' \int_{\mathbf{R}^2} v^+ \leq \int_{\mathbf{R}^2} \left(e^{p'v^+} - 1 \right) = \int_{\{v>0\}} e^{p'v} \leq \int_{\mathbf{R}^2} e^v \leq C.$$

We conclude

$$\left\| v_3^+ \right\|_{L^1(B_1)} \leq C$$

and, therefore,

$$\left\| v_3^+ \right\|_{L^\infty(B_{1/2})} \leq C.$$

Finally, in

$$-\Delta v = Ve^v = Ve^{v_1} \cdot e^{v_2+v_3} = g,$$

we have

$$\begin{aligned} & e^{v_2+v_3} \in L^\infty(B_{1/2}) \\ & V \in L^p(B_1) \\ & e^{v_1} \in L^{1/\varepsilon}(B_1) \end{aligned}$$

with $1/\varepsilon > p'$, and, therefore,

$$\|g\|_{L^{1+\delta}(B_{1/2})} \leq C \tag{2.217}$$

with $\delta > 0$. Then, the elliptic regularity guarantees the decomposition $v = w + h$ with

$$\|w\|_{L^\infty(B_{1/2})} \leq C \|g\|_{L^{1+\delta}(B_{1/2})}, \tag{2.218}$$

where h is harmonic in $B_{1/2}$, and, therefore, it holds that

$$\begin{aligned} \left\| h^+ \right\|_{L^\infty(B_{1/4})} & \leq C \left\| h^+ \right\|_{L^1(B_{1/2})} \\ & \leq C \left(\left\| v^+ \right\|_{L^1(B_{1/2})} + \|w\|_{L^1(B_{1/2})} \right) \\ & \leq C \end{aligned} \tag{2.219}$$

by the mean value theorem. We obtain

$$\left\| v^+ \right\|_{L^\infty(B_{1/4})} \leq C$$

with $C > 0$ independent of x_0 by (2.217)-(2.219), and the proof is complete. □

Similarly, we obtain the following theorem called the rough estimate in the previous paragraph.

Theorem 2.24. *Let $1 < p \leq \infty$, $c_1, c_2 > 0$, $0 < \varepsilon_0 < 4\pi/p'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ be given, $\Omega \subset \mathbf{R}^2$ a bounded domain, and $K \subset \Omega$ a compact set. Then, there exists $C > 0$ such that*

$$\begin{aligned} &-\Delta v = V(x)e^v \qquad \text{in } \Omega \\ &\|V\|_p \leq c_1, \quad \left\|v^+\right\|_1 \leq c_2, \quad \int_\Omega |V(x)|\, e^v \leq \varepsilon_0 \end{aligned}$$

implies

$$\left\|v^+\right\|_{L^\infty(K)} \leq C.$$

Proof: From the covering argument, we may assume $\Omega = B_R$ and $K = \overline{B_{R/2}}$, where $B_R = B(0,R)$. We decompose $v = v_1 + v_2$, where

$$-\Delta v_1 = V(x)e^v \quad \text{in } \Omega, \qquad v_1 = 0 \quad \text{on } \partial\Omega$$

and v_2 is harmonic in Ω. First, we apply the mean value theorem to v_2 and obtain

$$\left\|v_2^+\right\|_{L^\infty(B_{3R/4})} \leq C_R \left\|v_2^+\right\|_{L^1(B_R)} \leq C_R \left(\|v_1\|_{L^1(B_R)} + c_2 \right).$$

On the other hand, we have

$$\left\| e^{|v_1|} \right\|_{L^{p'+\delta}(B_R)} \leq C$$

by Theorem 2.22, where $\delta > 0$. Similarly to the proof of the previous theorem, this inequality implies

$$\left\|v_2^+\right\|_{L^\infty(B_{3R/4})} \leq C$$

and, consequently,

$$\|e^v\|_{L^{p'+\delta}(B_{3R/4})} \leq C.$$

Then, it holds that

$$\|Ve^v\|_{L^q(B_{3R/4})} \leq C$$

by the assumption. Using $v = w_1 + w_2$ with

$$-\Delta w_1 = V(x)e^v \quad \text{in } B_{3R/4}, \qquad w_1 = 0 \quad \text{on } \partial B_{3R/4},$$

and $\Delta w_2 = 0$ in $B_{3R/4}$, we obtain

$$\left\|v^+\right\|_{L^\infty(B_{R/2})} \leq C,$$

similarly, from the elliptic estimate and the mean value theorem.[123] The proof is complete. □

Now, we show Theorem 2.12 in the following extended form.

Theorem 2.25 ([32]). *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, and $v_k = v_k(x)$, $k = 1, 2, \dots$ satisfy*

$$-\Delta v_k = V_k(x) e^{v_k} \qquad \text{in } \Omega$$

for $V_k = V_k(x) \geq 0$, and assume the existence of $1 < p \leq \infty$ and $c_1, c_2 > 0$ such that

$$\|V_k\|_p \leq c_1, \qquad \|e^{v_k}\|_{p'} \leq c_2$$

for $k = 1, 2, \dots$. Then, passing to a sub-sequence, we have the following alternatives:

(i) *$\{v_k\}$ is locally uniformly bounded in Ω.*
(ii) *$v_k \to -\infty$ locally uniformly in Ω.*
(iii) *There is a finite set $\mathscr{S} = \{a_i\} \subset \Omega$ and $\alpha_i \geq 4\pi/p'$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathscr{S}$ and*

$$V_k(x) e^{v_k} dx \rightharpoonup \sum_i \alpha_i \delta_{a_i}(dx)$$

in $\mathscr{M}(\Omega)$. Furthermore, $\mathscr{S}$ is the blowup set of $\{v_k\}$ in Ω.

Proof: Since $\{V_k e^{v_k}\}_k$ is bounded in $L^1(\Omega)$, we obtain

$$V_k e^{v_k} dx \rightharpoonup \mu(dx)$$

in $\mathscr{M}(\Omega)$, passing to a sub-sequence. Then, the set

$$\Sigma = \left\{x_0 \mid \mu(\{x_0\}) \geq 4\pi/p'\right\}$$

is finite by $\mu(\Omega) \leq c_1 \cdot c_2$.
If $x_0 \notin \Sigma$, there is $\psi \in C_0^\infty(\Omega)$ satisfying $0 \leq \psi \leq 1$, $\psi = 1$ around x_0, and

$$\int_\Omega \psi d\mu < 4\pi/p'.$$

Then, we obtain $R_0 > 0$ such that

$$\left\|v_k^+\right\|_{L^\infty(B(x_0,R_0))} = O(1)$$

by Theorem 2.24. This conclusion means $\mathscr{S} \subset \Sigma$, while the converse is obvious. In fact, if $x_0 \notin \mathscr{S}$, then $\left\{v_k^+\right\}$ is uniformly bounded in $B(x_0, R_0)$ for some $R_0 > 0$, and this property implies $\mu(\{x_0\}) = 0$ and hence $x_0 \notin \Sigma$ by the elliptic regularity. Thus $\mu(\{x_0\}) < 4\pi/p'$ implies $x_0 \notin \mathscr{S}$. Now, we distinguish two cases.

Case 1: $\Sigma = \emptyset$ implies (i) or (ii).

Again, we may assume $\Omega = B_R$. In this case, $\left\{v_k^+\right\}$ is uniformly bounded in $\omega = B_{R/2}$. Defining w_k by

$$-\Delta w_k = f_k \quad \text{in } \omega, \qquad w_k = 0 \quad \text{on } \partial\omega$$

for $f_k = V_k e^{v_k}$, we see that $\{w_k\}$ is uniformly bounded in ω because $\{f_k\}$ is bounded in $L^p(\omega)$. Thus

$$\tilde{v}_k = v_k - w_k$$

is harmonic in ω and $\{\tilde{v}_k^+\}$ is uniformly bounded in ω. From the Harnack principle to the harmonic function, therefore, $\{\tilde{v}_k\}$ is locally uniformly bounded in ω, or otherwise $\tilde{v}_k \to -\infty$ locally uniformly in ω. These cases imply (i) and (ii), respectively.

Case 2: $\Sigma \neq \emptyset$ implies (iii).

Since Σ is finite, each $x_0 \in \Sigma$ admits $R > 0$ such that $\omega = B(x_0,R) \subset\subset \Omega$ and $B(x_0,R) \cap \Sigma = \{x_0\}$. From the above argument, $\{v_k\}$ is locally uniformly bounded or otherwise diverges to $-\infty$ locally uniformly, in $B(x_0,R) \setminus \{x_0\}$.
In the former case, it holds that

$$v_k \geq -C \qquad \text{on } \partial\omega.$$

Defining z_k by

$$-\Delta z_k = f_k \quad \text{in } \omega, \qquad z_k = 0 \quad \text{on } \partial\omega,$$

we obtain $v_k \geq z_k$ in ω. We have, on the other hand,

$$f_k(x)dx \rightharpoonup \alpha\delta_{x_0}(dx) + f(x)dx$$

in $\mathscr{M}(\overline{\omega})$ with

$$\begin{aligned} &\alpha \geq 4\pi/p' \\ &0 \leq f \in L^1(\omega), \end{aligned}$$

and, therefore, $z_k \to z$ locally uniformly in $\overline{\omega} \setminus \{x_0\}$ with

$$z(x) \geq \frac{4\pi}{p'} \cdot \frac{1}{2\pi} \log \frac{1}{|x - x_0|} - O(1), \qquad x \in \overline{\omega} \setminus \{x_0\}.$$

This property implies

$$\begin{aligned} +\infty = \int_{B(x_0,R)} e^{p'z} &\leq \liminf_k \int_{B(x_0,R)} e^{p'v_k} \\ &\leq \liminf_k \|e^{v_k}\|_{p'}^{p'} < +\infty \end{aligned}$$

by Fatou's lemma, a contradiction.
Thus we obtain $v_k \to -\infty$ locally uniformly in $B(x_0,R) \setminus \{x_0\}$, and

$$V_k e^{v_k} \to 0$$

in $L^p_{loc}(\Omega \setminus \Sigma)$ from the convering argument. This conclusion implies

$$\mu(dx) = \sum_i \alpha_i \delta_{a_i}(dx),$$

and the proof is complete. □

2.3.7. Entire Solution

Classification of the entire solution is an important ingredient of the blowup analysis, where the method of moving plane is useful. The results for

$$-\Delta v = v^p, \ \ v > 0 \qquad \text{in } \mathbf{R}^n$$

with $1 < p < \infty$ and $n \geq 3$ are described in §2.1.2. A refined approach guarantees also the two-dimensional version formulated as follows.[56]

Theorem 2.26. *If*

$$-\Delta v = e^v \quad \textit{in } \mathbf{R}^2, \qquad \int_{\mathbf{R}^2} e^v < +\infty, \tag{2.220}$$

then it holds that

$$v(x) = \log\left\{ \frac{8\mu^2}{\left(1 + \mu^2 |x - x_0|^2\right)^2} \right\}, \qquad x_0 \in \mathbf{R}^2, \ \mu > 0. \tag{2.221}$$

We begin with several lemmas for the proof.

Lemma 2.2. *If (2.220), then it holds that*

$$\int_{\mathbf{R}^2} e^v \geq 8\pi.$$

Proof: We note that $v = v(x)$ is real-anlaytic. In particular,

$$\int_{\Omega_t} e^v = -\int_{\Omega_t} \Delta v = \int_{\partial\Omega_t} |\nabla v|$$
$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{1}{|\nabla v|}$$

for each $t \in \mathbf{R}$, and, therefore,

$$\left(-\frac{d}{dt}|\Omega_t|\right) \cdot \int_{\Omega_t} e^v \geq |\partial\Omega_t|^2 \geq 4\pi |\Omega_t|,$$

where $\Omega_t = \left\{x \in \mathbf{R}^2 \mid v(x) > t\right\}$. This implies

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Omega_t} e^v\right)^2 &= 2\int_{\Omega_t} e^v \cdot \left(-\int_{\partial\Omega_t} \frac{e^v}{|\nabla v|}\right) \\ &= 2e^t \cdot \frac{d}{dt}|\Omega_t| \cdot \int_{\Omega_t} e^v \leq -8\pi e^t \cdot |\Omega_t|. \end{aligned}$$

Operating $\int_{-\infty}^{\infty} \cdot dt$, we obtain

$$\begin{aligned}
-\left(\int_{\mathbf{R}^2} e^v\right)^2 &\leq -8\pi \int_{-\infty}^{\infty} e^t\, |\Omega_t|\, dt = 8\pi \int_{-\infty}^{\infty} e^t \cdot \frac{d}{dt}\, |\Omega_t|\, dt \\
&= -8\pi \int_{-\infty}^{\infty} e^t dt \cdot \int_{\partial\Omega_t} \frac{1}{|\nabla v|} = 8\pi \int_{-\infty}^{\infty} \frac{d}{dt} \int_{\Omega_t} e^v \\
&= -8\pi \int_{\mathbf{R}^2} e^v.
\end{aligned}$$

The proof is complete. □

Lemma 2.3. *We have*

$$\frac{v(x)}{\log|x|} \to -\frac{1}{2\pi} \int_{\mathbf{R}^2} e^v \leq -4 \tag{2.222}$$

uniformly as $|x| \to \infty$.

Proof: We see that

$$w(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} (\log|x-y| - \log|y|)\, e^{v(y)} dy$$

is well-defined by $\int_{\mathbf{R}^2} e^v < +\infty$. Furthermore, it holds that

$$\begin{aligned}
&\Delta w = e^v \quad \text{in } \mathbf{R}^2 \\
&\frac{w(x)}{\log|x|} \to \frac{1}{2\pi} \int_{\mathbf{R}^2} e^v \qquad \text{as } |x| \to \infty
\end{aligned}$$

uniformly. Writing $u = v + w$, we have

$$\Delta u = 0, \qquad u(x) \leq C + C_1 \log(|x| + 1) \quad \text{in } \mathbf{R}^2$$

because $v(x)$ is bounded from above by Theorem 2.24. Then, Liouville's theorem assures that v is a constant, and the proof is complete. □

Method of the second moment described in §1.1.6 guarantees

$$\int_{\mathbf{R}^2} e^v \leq 8\pi, \tag{2.223}$$

see §2.4.2. The radial symmetry of $v = v(x)$, particularly the decreasing in r direction, however, is a key factor in the blowup argument developed in §2.3.8. The classification (2.221) is thus obtained by the radial symmetry of the solution,

$$v = v(|x - x_0|) \tag{2.224}$$

for some $x_0 \in \mathbf{R}^2$. This property is obtained if v has the axile symmetry of two **Q**-independent directions, and, therefore, is reduced to the symmetry of v in x_1-direction because of the rotation and translation invariance of (2.220). Once radial symmetry (2.224) is obtained, the proof of Theorem 2.26 is obvious. Thus (2.220) implies (2.221) and the proof will be complete.

To show the symmetry of v in x_1-dierction, we assume the maximum of v in $x_1 < -3$, regarding Lemma 2.3. Then, we put

$$\begin{aligned}
\Sigma_\lambda &= \{(x_1,x_2) \mid x_1 < \lambda\} \\
T_\lambda &= \partial\Sigma_\lambda = \{(x_1,x_2) \mid x_1 = \lambda\} \\
x^\lambda &= (2\lambda - x_1, x_2) \\
w_\lambda(x) &= v(x^\lambda) - v(x) \\
\overline{w}_\lambda(x) &= w_\lambda(x)/g(x)
\end{aligned}$$

for $\lambda \in \mathbf{R}$, where $g(x) = \log(|x| - 1)$. This $\overline{w}_\lambda(x)$ is well-defined for $x \in \Sigma_\lambda$ with $\lambda < -2$, and it follows that

$$\Delta w_\lambda = e^\psi w_\lambda \tag{2.225}$$

$$\Delta\overline{w}_\lambda + \frac{2}{g}\nabla g \cdot \nabla\overline{w}_\lambda + \left(\exp\psi + \frac{\Delta g}{g}\right)\overline{w}_\lambda = 0, \tag{2.226}$$

where $\psi(x)$ is between $v(x)$ and $v(x^\lambda)$.

Lemma 2.4. *There is $R_0 > 2$ independent of $\lambda < -2$ such that if x_0 attains the negative minimum of $\overline{w}_\lambda$ in Σ_λ, then $|x_0| < R_0$.*

Proof: Since

$$\frac{\Delta g}{g} = \frac{1}{|x|\,(|x|-1)^2 \log(|x|-1)}$$

we obtain $R_0 > 2$ such that

$$\exp v + \frac{\Delta g}{g} < 0$$

if $\mathbf{R}^2 \setminus B(0,R_0)$ by Lemma 2.3. From the assumption, it holds that

$$\psi(x_0) \le \max\left\{v(x_0), v(x_0^\lambda)\right\} = v(x_0),$$

and, therefore,

$$\exp\psi(x_0) + \frac{\Delta g}{g}(x_0) < 0$$

if $|x_0| \ge R_0$. This is impossible from (2.226) and the maximum principle, and hence $|x_0| < R_0$ follows. The proof is complete. □

To prove

$$v^{\lambda_0} \equiv v \tag{2.227}$$

for some λ_0, we note that

$$\overline{w}_\lambda \ge 0 \qquad \text{in } \Sigma_\lambda \tag{2.228}$$

holds for $\lambda < -R_0$. In fact, we have

$$\lim_{|x|\to\infty} \overline{w}_\lambda(x) = 0$$

by Lemma 2.3, and if this property is not the case, $\overline{w}_\lambda$ attains a negative minimum in Σ_λ, which is impossible by Lemma 2.4.
Since v attains the maximum in $x_1 < -3$, we obtain the maximum λ_0', denoted by λ_0, such that (2.228) holds for $\lambda \le \lambda_0'$. We show that this λ_0 satisfies the following.

(a) $w_\lambda > 0$ in Σ_λ for $\lambda < \lambda_0$ and $\frac{\partial v}{\partial x_1} > 0$ in Σ_{λ_0}.
(b) $w_{\lambda_0} \equiv 0$ in Σ_{λ_0}.

For this purpose, we use

$$\frac{\partial v}{\partial x_1} \ge 0, \qquad x_1 \le \lambda_0' \tag{2.229}$$

and

$$w_\lambda \equiv 0 \quad \Leftrightarrow \quad \frac{\partial v}{\partial x_1} = 0 \ \text{ somewhere on } T_\lambda \tag{2.230}$$

for $\lambda \le \lambda_0'$, the latter being obtained by the strong maximum principle and the Hopf lemma. To prove (a), first, we show $w_\lambda > 0$ in Σ_λ if $\lambda < \lambda_0$. In fact, if this property is not the case, it holds that

$$w_{\lambda_0-\delta} \equiv 0$$

for $\delta > 0$ from the strong maximum principle, and, therefore,

$$v(\lambda_0 - 2\delta, x_2) = v(\lambda_0, x_2).$$

This conclusion implies

$$\frac{\partial v}{\partial x_1} = 0, \qquad \lambda_0 - 2\delta \le x_1 \le \lambda_2$$

by (2.229). In particular, we obtain

$$\frac{\partial v}{\partial x_1} = 0 \qquad \text{on } T_{\lambda_0-\delta}$$

and hence

$$w_{\lambda_0-2\delta} \equiv 0$$

by (2.230). Repeating the argument, we obtain the independence of v in x_1, a contradiction.
Once this property is proven, it holds that

$$\frac{\partial w_\lambda}{\partial x_1} < 0 \qquad \text{on } T_\lambda$$

by the Hopf lemma, where $\lambda < \lambda_0$. This conclusion implies

$$\frac{\partial v}{\partial x_1} > 0, \qquad x_1 < \lambda_0,$$

and hence $\frac{\partial v}{\partial x_1} > 0$ in Σ_{λ_0}.
We have proven that v is strictly increasing in $x_1 < \lambda_0$, and hence $\lambda_0 < -3$. We can define $\overline{w}_\lambda$ in Σ_λ for $\lambda < \lambda_0 + 1$. We have also proven $\overline{w}_{\lambda_0} \geq 0$, and, therefore, in case $\overline{w}_{\lambda_0} \not\equiv 0$, it holds that

$$\overline{w}_{\lambda_0} > 0 \quad \text{in } \Sigma_{\lambda_0}, \qquad \frac{\partial \overline{w}_{\lambda_0}}{\partial x_1} < 0 \quad \text{on } T_{\lambda_0} \tag{2.231}$$

by (2.226) from the strong maximum principle and the Hopf lemma. On the other hand, there is a sequence $\lambda_k \downarrow \lambda_0$ and $x_k' \in \Sigma_{\lambda_k}$ such that

$$\overline{w}_{\lambda_k}(x_k') < 0$$

from the definition of λ_0. This property assures the minimum point of $\overline{w}_{\lambda_k}$ in Σ_{λ_k}, denoted by $x_k \in \Sigma_{\lambda_k}$, by Lemma 2.3.
We obtain

$$|x_k| \leq R_0$$

by Lemma 2.4, and passing to a subsequence, $x_k \to x_0$ with

$$\begin{aligned} & x_0 \in \Sigma_{\lambda_0} \cup T_{\lambda_0} \\ & \overline{w}_{\lambda_0}(x_0) \leq 0 \\ & \nabla \overline{w}_{\lambda_0}(x_0) = 0, \end{aligned}$$

a contradiction to (2.231). □

2.3.8. Blowup Analysis

This paragraph is devoted to the proof of the quantized blowup mechanism to (2.215), namely, Theorem 2.13. Henceforth, k is sufficiently large, and the subsequence of $\{v_k\}$ is denoted by the same symbol. The fundamental idea of the blowup analysis is used in the proof of the following lemma.

Lemma 2.5. *Under the assumption of Theorem 2.13, it holds that*

$$V(0) > 0.$$

Proof: We obtain $x_k \in B$ such that

$$\begin{aligned} & v_k(x_k) = \max_{\overline{B}} v_k \\ & x_k \to 0 \\ & v_k(x_k) \to +\infty. \end{aligned}$$

This relation implies

$$\delta_k = e^{-v_k(x_k)/2} \to 0$$

and

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) + 2\log\delta_k$$

is defined in $B(0, R/(2\delta_k))$, which satisfies

$$\begin{aligned} & -\Delta\tilde{v}_k = V_k(\delta_k x + x_k)e^{\tilde{v}_k} \\ & \tilde{v}_k \leq 0 = \tilde{v}_k(0) \qquad \text{in } B(0, R/(2\delta_k)) \\ & \int_{B(0,R/(2\delta_k))} e^{\tilde{v}_k} \leq C_0. \end{aligned}$$

Given $r > 0$, we obtain the well-definedness of $\{\tilde{v}_k\}$ in $B_r = B(0,r)$, and Theorem 2.25 is applicable to $\Omega = B_r$. The other cases than (i) are impossible, and, therefore, $\{\tilde{v}_k\}$ is uniformly bounded in $B(0,r)$. The standard diagonal argument and the elliptic estimate guarantee the convergence

$$\tilde{v}_k \to \tilde{v} \qquad \text{in } C^{1,\alpha}_{loc}(\mathbf{R}^2),$$

where $0 < \alpha < 1$, and it holds that

$$\begin{aligned} & -\Delta\tilde{v} = V(0)e^{\tilde{v}}, \;\; \tilde{v} \leq 0 = \tilde{v}(0) \qquad \text{in } \mathbf{R}^2 \\ & \int_{\mathbf{R}^2} e^{\tilde{v}} \leq C_0. \end{aligned}$$

In case $V(0) = 0$, $\tilde{v}$ is a non-negative harmonic function in $\mathbf{R}^2$, and hence is a constant by Liouville's theorem. This property implies

$$\int_{\mathbf{R}^2} e^{\tilde{v}} = +\infty,$$

a contradiction. □

In the above lemma, we have

$$\begin{aligned} & \tilde{v}(x) = \log\frac{1}{\left(1 + \frac{V(0)}{8}|x|^2\right)^2} \\ & \int_{\mathbf{R}^2} V(0)e^{\tilde{v}} = 8\pi \end{aligned}$$

by Theorem 2.26. In fact, $v(x) = \tilde{v}(x) + \log V(0)$ satisfies

$$\begin{aligned} & -\Delta v = e^v \quad \text{in } \mathbf{R}^2, \qquad \int_{\mathbf{R}^2} e^v < +\infty \\ & \max_{\mathbf{R}^2} v = v(0) = \log V(0). \end{aligned}$$

Taking smaller $R > 0$, we may assume

$$a \leq V_k(x) \leq b, \qquad x \in B \tag{2.232}$$

for $a, b > 0$. This condition makes it possible to apply sup + inf inequality, which takes the role of the Harnack inequality or the monotonicity formula. Then, we can exhaust the blowup mechanism by counting the particle concentrations.

Lemma 2.6. *Given $a,b>0$, we obtain $c_1\geq 1$, $c_2>0$ (independent of $R>0$) such that*

$$-\Delta v = V(x)e^v,\ \ a\leq V(x)\leq b \qquad \text{in } B_R$$

implies

$$v(0)+c_1\inf_{B_r} v+2(c_1+1)\log r\leq c_2$$

for $0<r\leq R$.

Proof: The assertion is an immediate consequence of the sup + inf inequality and the rescaling. In fact, we take

$$\tilde{v}(x)=v(rx)+2\log r$$

for $r\in(0,R]$, and obtain

$$-\Delta\tilde{v}=V(rx)e^{\tilde{v}} \qquad \text{in } B_1.$$

Theorem 2.21 applied to $\Omega=B_1$ and $K=\{0\}$, thus implies

$$\tilde{v}(0)+c_1\inf_{B_1}\tilde{v}\leq c_2$$

or, equivalently,

$$v(0)+2(c_1+1)\log r+c_1\inf_{B_r} v\leq c_2.$$

Here, we have

$$\inf_{B_r} v=\inf_{\partial B_r} v$$

because v is super-harmonic, and the proof is complete. □

We use the following lemma also, obtained by the Harnack inequality for harmonic functions.

Lemma 2.7. *There is $\beta\in(0,1)$ such that given $C_1,C_2>0$, we have $c_3>0$ (independent of $R,R_0>0$ in $0<R_0\leq R/4$) such that*

$$\begin{aligned}&-\Delta v=V(x)e^v\\&|V(x)|\leq C_1\\&v(x)+2\log|x|\leq C_2 \qquad \text{in } B_R\setminus\overline{B_{R_0}}\end{aligned}$$

implies

$$\sup_{\partial B_r} v\leq c_3+\beta\inf_{B_r} v+2(\beta-1)\log r$$

for $2R_0\leq r\leq R/2$.

Proof: Taking $r \in [2R_0, R/2]$, we put

$$\tilde{v}(x) = v(rx) + 2\log r$$

and obtain

$$\begin{aligned}
&-\Delta\tilde{v} = V(rx)e^{\tilde{v}} \\
&\tilde{v}(x) = v(rx) + 2\log(r|x|) - 2\log|x| \leq C_2 + 2\log 2 \\
&\left|V(rx)e^{\tilde{v}}\right| \leq C_1 \exp(C_2 + 2\log 2) \qquad \text{in } B_2 \setminus \overline{B_{1/2}}.
\end{aligned}$$

Then, we have C_4 determined by C_1, C_2 such that

$$|w| \leq C_4 \qquad \text{in } B_2 \setminus \overline{B_{1/2}},$$

where

$$\begin{aligned}
-\Delta w &= V(rx)e^{\tilde{v}} \quad \text{in } B_2 \setminus \overline{B_{1/2}} \\
w &= 0 \qquad\qquad \text{on } \partial(B_2 \setminus \overline{B_{1/2}}).
\end{aligned}$$

The Harnack inequality is valid to the non-negative harmonic function

$$h = w - \tilde{v} + C_5$$

with $C_5 = C_4 + C_2 + 2\log 2$, we obtain the absolute constant $\beta \in (0,1)$ such that

$$\beta \sup_{\partial B_1} \leq \inf_{\partial B_1} h.$$

The right-hand and the left-hand sides are estimated from above and from below by

$$C_5 + C_4 - \sup_{\partial B_1} \tilde{v} = C_5 + C_4 - 2\log r - \sup_{\partial B_r} v$$

and

$$\beta\left(C_5 - C_4 - 2\log r - \inf_{\partial B_r} v\right),$$

respectively. Then, we obtain

$$\sup_{\partial B_r} v \leq (1-\beta)C_5 + (1-\beta)C_4 + 2(\beta - 1)\log r + \beta \inf_{\partial B_r} v$$

and the proof is complete. □

We are ready to prove the key inequality.[181]

Theorem 2.27. *Given $a, b > 0$, $C_1 > 0$, we obtain $\gamma > 0$, $C_2 > 0$ independent of $0 < R_0 \leq R/4$ such that*

$$\begin{aligned}
&-\Delta v = V(x)e^v, \;\; a \leq V(x) \leq b \qquad \textit{in } B_R \\
&v(x) + 2\log|x| \leq C_1 \qquad\qquad\quad \textit{in } B_R \setminus \overline{B_{R_0}}
\end{aligned} \tag{2.233}$$

implies

$$e^{v(x)} \leq C_2 e^{-\gamma v(0)} \cdot |x|^{-2(\gamma+1)} \tag{2.234}$$

for $2R_0 \leq |x| \leq R/2$.

Proof: Lemmas 2.6 and 2.7 guarantee

$$\inf_{\partial B_r} v \le \frac{c_2}{c_1} - \frac{1}{c_1} v(0) - 2(1+\frac{1}{c_1}) \log r$$

for $0 < r \le R$ and

$$\sup_{\partial B_r} v \le \left(c_3 + \beta \cdot \frac{c_2}{c_1}\right) - \frac{\beta}{c_1} v(0) - 2\left(\frac{\beta}{c_1} + 1\right) \log r$$

for $2R_0 < r \le R/2$, respectively, and, therefore, (2.234) holds for

$$\gamma = \beta / c_1$$
$$C_2 = \exp\left(c_3 + \beta \cdot \frac{c_2}{c_1}\right).$$

The proof is complete. □

Here, we give the estimate from below in Theorem 2.13.

Lemma 2.8. *Under the assumption of Theorem 2.13, passing to a subsequence, we have*

$$m \in \mathbf{N}, \quad 1 \le m \le V(0) \cdot C_0 / (8\pi)$$
$$\left\{x_k^j\right\} \subset B_R, \quad \lim_{k\to\infty} x_k^j = 0, \qquad 0 \le j \le m-1$$
$$\left\{\sigma_k^j\right\}, \quad \lim_{k\to\infty} \sigma_k^j = +\infty, \qquad 0 \le j \le m-1$$

such that

$$v_k(x_k^j) = \max_{B_{\sigma_k^j \delta_k^j}(x_k^j)} v_k \to +\infty \tag{2.235}$$

$$B_{2\sigma_k^j \delta_k^j}(x_k^j) \cap B_{2\sigma_k^i \delta_k^i}(x_k^i) = \emptyset, \qquad i \ne j \tag{2.236}$$

$$\left.\frac{\partial}{\partial t} v_k(tx + x_k^j)\right|_{t=1} < 0, \qquad \delta_k^j \le |x| \le 2\sigma_k^j \delta_k^j \tag{2.237}$$

$$\lim_k \int_{B_{2\sigma_k^j \delta_k^j}(x_k^j)} V_k e^{v_k} = \lim_k \int_{B_{\sigma_k^j \delta_k^j}(x_k^j)} V_k e^{v_k} = 8\pi \tag{2.238}$$

$$\max_{x \in \overline{B_R}} \left\{ v_k(x) + 2 \log \min_j \left|x - x_k^j\right| \right\} \le C, \tag{2.239}$$

where $\delta_k^j = e^{-v_k(x_k^j)/2}$.

Proof: We take x_k in the proof of Lemma 2.5, denoted by x_k^0. Thus it holds that

$$x_k^0 \in B_R$$
$$v_k(x_k^0) = \max_{\overline{B_R}} v_k$$
$$x_k^0 \to 0$$
$$v_k(x_k^0) \to +\infty$$
$$\tilde{v}_k^0 \to \tilde{v} \qquad \text{in } C^{1,\alpha}_{loc}(\mathbf{R}^2)$$

for $0 < \alpha < 1$ and

$$\tilde{v}_k^0(x) = v_k(\delta_k^0 x + x_k^0) + 2\log\delta_k^0$$
$$\tilde{v}(x) = \log\frac{1}{\left(1 + \frac{V(0)}{8}|x|^2\right)^2}.$$

From the diagonal argument, we may assume $\sigma_k^0 \to +\infty$ satisfying

$$\left\|\tilde{v}^0 - \tilde{v}\right\|_{C^{1,\alpha}\left(B_{2\sigma_k^0}\right)} \to 0.$$

This relation implies

$$\int_{B_{2\sigma_k^0\delta_k^0}(x_k^0)} V_k e^{v_k} = \int_{B_{2\sigma_k^0}} V_k(\delta_k^0\cdot + x_k^0)e^{\tilde{v}_k^0} \to 8\pi$$
$$\int_{B_{\sigma_k^0\delta_k^0}(x_k^0)} V_k e^{v_k} = \int_{B_{\sigma_k^0}} V_k(\delta_k^0\cdot + x_k^0)e^{\tilde{v}_k^0} \to 8\pi$$
$$\left.\frac{\partial}{\partial t}v_k(tx + x_k^0)\right|_{t=1} < 0, \qquad \delta_k^0 \le |x| \le 2\sigma_k^0\delta_k^0.$$

Thus we obtain (2.235), (2.237), (2.238) for $m = 0$.
Now, we assume (2.235), (2.236), (2.237), (2.238) for $m = \ell$. If (2.239) holds, then the proof is complete. Otherwise, we obtain

$$M_k \equiv \max_{x\in\overline{B_R}}\left\{v_k(x) + 2\log\min_{0\le j\le \ell-1}\left|x - x_k^j\right|\right\} \to +\infty, \tag{2.240}$$

passing to a subsequence. Then, this maximum M_k is attained by some $\overline{x}_k^\ell \in B_R$, and it holds that $v_k(\overline{x}_k^\ell) \to +\infty$. Thus

$$\overline{x}_k^\ell \to 0$$
$$\overline{\delta}_k^\ell \equiv e^{-v_k(\overline{x}_k^\ell)/2} \to 0$$

and (2.240) reads

$$m_k \equiv \min_{0\le j\le \ell-1}\frac{\left|\overline{x}_k^\ell - x_k^j\right|}{\overline{\delta}_k^\ell} \to +\infty. \tag{2.241}$$

For $|x| \le m_k/2$, it holds that

$$\min_{0\le j\le \ell-1}\left|\overline{x}_k^\ell + \overline{\delta}_k^\ell x - x_k^j\right| \ge \min_{0\le j\le \ell-1}\left|\overline{x}_k^\ell - x_k^j\right| - \overline{\delta}_k^\ell|x|$$
$$\ge \frac{1}{2}\min_{0\le j\le \ell-1}\left|\overline{x}_k^\ell - x_k^j\right|.$$

We have also

$$v_k(x) + 2\log\min_{0\le j\le \ell-1}\left|x - x_k^j\right| \le v_k(\overline{x}_k^\ell) + 2\log\min_{0\le j\le \ell-1}\left|\overline{x}_k^\ell - x_k^j\right|$$

and, therefore,

$$
\begin{aligned}
&\overline{v}_k(x) \equiv v_k(\overline{\delta}_k^\ell x + \overline{x}_k^\ell) + 2\log \overline{\delta}_k^\ell \\
&\leq v_k(\overline{x}_k^\ell) + 2\log \min_{0\leq j\leq \ell-1} \left|\overline{x}_k^\ell - x_k^j\right| + 2\log \overline{\delta}_k^\ell \\
&\qquad -2\log \min_{0\leq j\leq \ell-1} \left|\overline{\delta}_k^\ell x + \overline{x}_k^\ell - x_k^j\right| \\
&\leq v_k(\overline{x}_k^\ell) + 2\log \min_{0\leq j\leq \ell-1} \left|\overline{x}_k^\ell - x_k^j\right| + 2\log \overline{\delta}_k^\ell - 2\log \frac{1}{2} \min_{0\leq j\leq \ell-1} \left|\overline{x}_k^\ell - x_k^j\right| \\
&= 2\log 2.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&-\Delta \overline{v}_k = V_k(\overline{\delta}_k^\ell x + \overline{x}_k^\ell) e^{\overline{v}_k} \\
&\overline{v}_k(x) \leq 2\log 2 \qquad\qquad \text{in } B_{m_k/2} \\
&\overline{v}_k(0) = 0
\end{aligned}
$$

and, therefore, Theorems 2.26 and 2.25 guarantees $\overline{a} > 0$ and $\overline{x} \in \mathbf{R}^2$ such that

$$
\begin{aligned}
&\overline{v}_k \to \overline{v} \qquad \text{in } C^{1,\alpha}_{loc}(\mathbf{R}^2) \\
&\overline{v}(x) = \log \frac{\overline{a}^2}{\left(1 + \mu^2 \overline{a}^2 \left|x - \overline{x}\right|^2\right)^2} \\
&\overline{v}(0) = 0 \\
&\overline{v} \leq 2\log 2 \qquad \text{in } \mathbf{R}^2,
\end{aligned}
$$

where $\mu = (V(0)/8)^{1/2}$. In particular, we obtain

$$
\begin{aligned}
&1 \leq \overline{a} \leq 2 \\
&|\overline{x}| \leq 1/(2\mu),
\end{aligned} \tag{2.242}
$$

and now define

$$
L = \max_{|x|\leq 1/\mu} \overline{v}(x) - \min_{|x|\leq 1/\mu} \overline{v}(x).
$$

From the diagonal argument, we may assume $\sigma_k^\ell \to +\infty$ satisfying

$$
\|\overline{v}_k - \overline{v}\|_{C^{1,\alpha}(B_{4\sigma_k^\ell})} \to 0
$$

and, therefore,

$$
\left.\frac{\partial}{\partial t} \overline{v}_k(ty + \overline{x})\right|_{t=1} < 0, \qquad \frac{1}{2} \leq |y| \leq 4\sigma_k^\ell.
$$

This condition assures $y_k^\ell \in B_1$ satisfying

$$
\overline{v}_k(y_k^\ell + \overline{x}) = \max_{y\in \overline{B_{4\sigma_k^\ell}}} \overline{v}_k(y + \overline{x}),
$$

and it holds that

$$x_k^\ell \equiv \overline{\delta}_k^\ell (y_k^\ell + \overline{x}) + \overline{x}_k^\ell \to 0$$

by (2.242), and furthermore,

$$v_k(\overline{x}_k^\ell) \le v_k(x_k^\ell) \le v_k(\overline{x}_k^\ell) + L.$$

Thus defining

$$\begin{aligned} \tilde{v}_k^\ell(x) &= v_k(\delta_k^\ell x + x_k^\ell) + 2\log \delta_k^\ell \\ \delta_k^\ell &= e^{-v_k(x_k^\ell)/2} \to 0, \end{aligned}$$

we obtain

$$\begin{aligned} &\delta_k^\ell \le \overline{\delta}_k^\ell \le e^{L/2} \cdot \delta_k^\ell \\ &v_k(x_k^\ell) = \max_{|x - x_k^\ell| \le 4\sigma_k^\ell \delta_k^\ell} v_k(x) \to +\infty, \end{aligned}$$

and, therefore, Theorems 2.26 and 2.25 guarantee

$$\begin{aligned} &\int_{B_{2\sigma_k^\ell \delta_k^\ell}(x_k^\ell)} V_k e^{v_k} = \int_{B_{2\sigma_k^\ell}} V_k(\delta_k^\ell \cdot + x_k^\ell) e^{\tilde{v}_k^\ell} \to 8\pi \\ &\int_{B_{\sigma_k^\ell \delta_k^\ell}(x_k^\ell)} V_k e^{v_k} = \int_{B_{\sigma_k^\ell}} V_k(\delta_k^\ell \cdot + x_k^\ell) e^{\tilde{v}_k^\ell} \to 8\pi \\ &\left. \frac{\partial}{\partial t} v_k(tx + x_k^\ell) \right|_{t=1} < 0, \qquad \delta_k^\ell \le |x| \le 2\sigma_k^\ell \delta_k^\ell, \end{aligned}$$

similarly. This relation, combined with (2.241), implies (2.235)-(2.238) for $m = \ell + 1$, and we can continue this process as far as (2.239) fails. However, the number of this process is bounded from above by $V(0)C_0/(8\pi)$, and the proof is complete. □

The proof of Theorem 2.13 is reduced to

$$\lim_k \int_{B_R \setminus \bigcup_\ell B_{\sigma_k^\ell \delta_k^\ell}(x_k^\ell)} V_k e^{v_k} = 0 \tag{2.243}$$

by Lemma 2.8, which is proven by Theorem 2.27 and (2.239). It is obvious that (2.243) is reduced to the following lemma by Lemma 2.8.

Lemma 2.9. *Under the assumption of Theorem 2.18, if we have $m \geq 1$, $\left\{x_k^j\right\} \subset B_R$, and $\left\{r_k^j\right\} \subset (0,\infty)$ satisfying*

$$x_k^j \to 0, \quad v_k(x_k^j) \to +\infty$$

$$\lim_{k\to\infty} \frac{r_k^j}{\delta_k^j} = +\infty \tag{2.244}$$

$$B_{r_k^j}(x_k^j) \cap B_{r_k^j}(x_k^i) = \emptyset, \qquad i \neq j$$

$$\max\left\{v_k(x) + 2\log \min_{0\leq j\leq m-1} \left|x - x_k^j\right| \mid x \in \overline{B_R} \setminus \cup_{j=0}^{m-1} B_{r_k^j}(x_k^j)\right\} \leq C \tag{2.245}$$

$$\lim_{k\to\infty} \int_{B_{2r_k^j}(x_k^j)} V_k e^{v_k} = \lim_k \int_{B_{r_k^j}(x_k^j)} V_k e^{v_k} = \beta_j \geq 4\pi, \tag{2.246}$$

then it holds that

$$\lim_{k\to\infty} \int_{B_R} V_k e^{v_k} = \sum_{j=0}^{m-1} \beta_j,$$

where $\delta_k^j = e^{-v_k(x_k^j)/2}$.

Proof: In the case of $m = 1$, we may assume that

$$x_k^0 = 0$$
$$\lim_{k\to\infty} r_k^0 = 0.$$

In fact, the lemma is obvious if the latter condition does not hold for any subsequence. Then, Theorem 2.27 is applicable by (2.245), and it holds that

$$e^{v_k(x)} \leq C_2 \cdot (\delta_k^0)^{2\gamma} \cdot |x|^{-2(\gamma+1)}, \qquad 2r_k^0 \leq |x| \leq R/2.$$

This inequality implies

$$\int_{B_{R/2}\setminus B_{2r_k^0}} V_k e^{v_k} \leq b \cdot C_2 \cdot (\delta_k^0)^{2\gamma} \cdot 2\pi \cdot \int_{2r_k^0}^{\infty} r^{-2(\gamma+1)} \cdot r dr$$
$$= C_3 \left(\frac{\delta_k^0}{2r_k^0}\right)^{2\gamma} \to 0$$

by (2.244). Then, we obtain

$$\lim_{k\to\infty} \int_{B_R} V_k e^{v_k} = \beta_0$$

by (2.246) and the assumption.
Supposing the lemma for $1, \dots, m-1$ with $m \geq 2$, we show it for m. Here, we may assume that

$$d_k \equiv \left|x_k^1 - x_k^0\right| = \min\left\{\left|x_k^i - x_k^j\right| \mid i \neq j\right\}$$

and $x_k^0 = 0$.

Case 1. $\left|x_k^i - x_k^j\right| \le Ad_k$, $0 \le i, j \le m-1$, for some $A \ge 1$.

In this case it holds that

$$\lim_{k\to\infty} \int_{B_{4Ad_k}} V_k e^{v_k} = \lim_{k\to\infty} \int_{B_{2Ad_k}} V_k e^{v_k} = \sum_{j=0}^{m-1} \beta_j. \tag{2.247}$$

Then, we apply the lemma for $m = 0$, putting

$$\begin{aligned}(r_k^0)' &= 2Ad_k \\ (\beta_0)' &= \sum_{j=0}^{m-1} \beta_j \\ (x_k^0)' &= 0,\end{aligned}$$

and obtain

$$\lim_{k\to\infty} \int_{B_R} V_k e^{v_k} = \sum_{j=0}^{m-1} \beta_j.$$

To prove (2.247), we take

$$\tilde{v}_k(x) = v_k(d_k x) + 2\log d_k$$

defined for $|x| \le R/d_k$, and put

$$\begin{aligned}&\tilde{V}_k(x) = V_k(d_k x) \\ &\tilde{x}_k^j = x_k^j/d_k, \quad \tilde{\delta}_k^j = e^{-\tilde{v}_k(\tilde{x}_k^j)/2} = \frac{\delta_k^j}{d_k}, \quad \tilde{r}_k^j = \frac{r_k^j}{d_k}\end{aligned}$$

for $0 \le j \le m-1$. It holds that $\tilde{x}_k^0 = 0$ and $d_k \to 0$. From the assumption, we have

$$\begin{aligned}&\frac{\tilde{r}_k^j}{\tilde{\delta}_k^j} = \frac{r_k^j}{\delta_k^j} \to +\infty \\ &B_{\tilde{r}_k^i}(\tilde{x}_k^i) \cap B_{\tilde{r}_k^j}(\tilde{x}_k^j) = \emptyset, \qquad i \ne j \\ &\max\left\{\tilde{v}_k(x) + 2\log \min_{0\le j\le m-1} \left|x - \tilde{x}_k^j\right| \mid x \in \overline{B_{R/d_k}} \setminus \cup_{j=0}^{m-1} B_{\tilde{r}_j}(\tilde{x}_k^j)\right\} \\ &\quad \le C \qquad (2.248) \\ &\lim_k \int_{B_{2\tilde{r}_k^j}(x_k^j)} \tilde{V}_k e^{\tilde{v}_k} = \lim_k \int_{B_{\tilde{r}_k^j}(\tilde{x}_k^j)} \tilde{V}_k e^{\tilde{v}_k} = \beta_j \qquad (2.249)\end{aligned}$$

for $0 \le j \le m-1$. We can assume, furthermore,

$$\tilde{x}_k^j \to \tilde{x}^j, \qquad 0 \le j \le m-1$$

by $\left|\tilde{x}_k^j\right| \le A$.

First, we obtain

$$\begin{aligned} &\tilde{v}_k \to -\infty \qquad \text{locally uniformly in } \mathbf{R}^2 \setminus \bigcup_{j=0}^{m-1} \left\{\tilde{x}_k^j\right\} \\ &1 \leq \left|\tilde{x}^i - \tilde{x}^j\right| \leq A, \qquad i \neq j \end{aligned} \tag{2.250}$$

by (2.248) and Theorem 2.25. If $\lim_{k\to\infty} \tilde{r}_k^j > 0$ holds to a subsequence, then it holds that

$$\int_{B_{1/2}(\tilde{x}^j)} \tilde{V}_k e^{\tilde{v}_k} \to \beta_j \tag{2.251}$$

by (2.249)-(2.250). In the other case of

$$\lim_{k\to\infty} \tilde{r}_k^j = 0,$$

we apply the lemma with $m = 0$. Then, (2.251) follows again.
By (2.250)-(2.251), it holds that

$$\lim_{k\to\infty} \int_{B_{4A}} \tilde{V}_k e^{\tilde{v}_k} = \lim_{k\to\infty} \int_{B_{2A}} \tilde{V}_k e^{\tilde{v}_k} = \sum_{j=0}^{m-1} \beta_j,$$

which means (2.247).

Case 2. The other case.

We obtain non-void $J_\ell \subset \{0,1,\ldots,m-1\}$, $\ell = 1,\ldots,k$, such that

$$\begin{aligned} &J_\ell \cap J_m = \emptyset, && \ell \neq m \\ &\bigcup_{\ell=1}^{m} J_\ell = \{0,1,\ldots,m-1\} \\ &\left|x_k^j - x_k^i\right| \leq A d_k, && i,j \in J_\ell \\ &\lim_{k\to\infty} \left|x_k^i - x_k^j\right| / d_k = \infty, && i \in J_\ell,\ j \in J_m,\ \ell \neq m. \end{aligned}$$

Taking $x_\ell \in J_\ell$, we obtain

$$\lim_{k\to\infty} \int_{B_{2Ad_k}(x_\ell)} V_k e^{v_k} = \lim_{k\to\infty} \int_{B_{Ad_k}(x_\ell)} V_k e^{v_k} = \sum_{j\in J_\ell} \beta_j$$

similarly to the case 1. On the other hand, we can apply the lemma for $m = k$:

$$\lim_{k\to\infty} \int_{B_R} V_k e^{v_k} = \sum_{\ell=1}^{k} \sum_{j=J_\ell} \beta_j = \sum_{j=0}^{m-1} \beta_j.$$

The proof is complete. □

2.3.9. Summary

We studied two-dimensional stationary mass quantization in detail.

(1) A typical profile of self-assembly, the quantized blowup mechanism, is observed in the stationary state of self-gravitating particles, the mean field of turbulence, and particularly, in the self-dual gauge field.
(2) This profile is controlled by the exponential nonlinearity competing two-dimensional diffusion.
(3) This equation is provided with several mathematical structures, complex function theory, theory of surfaces, real analysis, calculus of variation, spectral analysis, and particularly, blowup analysis.
(4) Blowup analysis is based on the self-similarity of the problem, and uses hierarchical arguments.
(5) For this purpose, it is necessary to apply the methods of second moment, moving planes, or the Harnack inequality to envelope the blowup mechanism.

2.4. Higher-Dimensional Blowup

Although the detailed study of non-stationary quantization is a project for the future, mass and energy quantizations are certainly observed in higher-space dimension. The methods of scaling and duality still work, in spite of unanswered questions. The last section of this book describes aspects of these unanswered questions such as the energy quantization in the semilinear parabolic equation, the method of duality applied to higher-dimensional problems, the higher-dimensional stationary mass quantization arising in the self-gravitating fluid, and the general dimension control of the blowup set. We thus end up with new observations to the fundamental equations of physics from the viewpoints of calculus of variation and a mean field hierarchy.

2.4.1. Semilinear Parabolic Equation

The semilinear parabolic equation with power nonlinearity

$$u_t - \Delta u = u^p, \ \ u > 0 \qquad \text{in } \mathbf{R}^n \times (0,T) \tag{2.252}$$

was introduce as a ”toy“ model of the incompressible Navier-Stokes equation

$$\begin{aligned} &v_t + (v\cdot\nabla)v - \Delta v = -\nabla p \\ &\nabla\cdot v = 0 \qquad \text{in } \mathbf{R}^n \times (0,T), \end{aligned} \tag{2.253}$$

see for example,[110,111] concerning the application of the hierachical argument to (2.253). Based on a variety of mathematical backgrounds, *scaling invariance*, *variational structure*, *comparison principle*, ...; however, the solution is provided with fruitful features observed widely in nonlinear problems; *critical exponents*, *blowup in finite time* or *blowup in infinite time*, *blowup rate*, *blowup profile*, the *complete blowup* in accordance with the *post-blowup continuation*, and so forth.[263,268]

The Cauchy problem to (2.252) is well-posed locally in time under the appropriate decay of u at $x=\infty$, and $T=T_{\max}\in(0,+\infty]$ denotes the maximum existence time. The case $T<+\infty$ is called the *blowup of the solution* because then it follows that

$$\lim_{t\uparrow T}\|u(\cdot,t)\|_\infty=+\infty.$$

In 1966, Fujita[105] formulated his exponent in comparison with the ODE part

$$u_t=u^p,\qquad u>0,$$

which is summarized as follows, see[137,171] for the critical case $p=1+\frac{2}{n}$.

(1) If $1<p\leq 1+\frac{2}{n}$, then any non-trivial solution blows-up in finite time.
(2) If $p>1+\frac{2}{n}$, then there is a non-trivial solution with $T=+\infty$.

Thus, the exponent $p_f=1+\frac{2}{n}$ is critical in (2.252) in accordance with the ODE part, which is dominant in the sub-critical nonlinearity of $1<p\leq p_f$.

Forward Self-Similar Transformation

The scaling invariance (2.163) to

$$u_t-\Delta u=u^p,\ \ u>0\qquad \text{in } \mathbf{R}^n\times(0,+\infty) \tag{2.254}$$

induces the *forward self-similar transformation*

$$\begin{aligned} v(y,s)&=(t+1)^{\frac{1}{p-1}}u(x,t)\\ y&=x/(t+1)^{1/2}\\ s&=\log(t+1),\end{aligned}$$

and then it follows that

$$\begin{aligned} &v_s-\Delta v=\frac{y}{2}\cdot\nabla v+\frac{v}{p-1}+v^p\\ &v>0 \qquad\qquad\qquad\qquad \text{in } \mathbf{R}^n\times(0,+\infty).\end{aligned} \tag{2.255}$$

The stationary solution to the rescaled equation (2.255) is called the *self-similar solution*. If $1+\frac{n}{2}<p<\frac{n+2}{n-2}$, the solution to (2.254) blows-up in finite time or converges to 0 in infinite time with the rate either of the Gauss kernerl $O(t^{-\frac{N}{2}})$ or that of the self-similar solution $O(t^{-\frac{1}{p-1}})$. The latter case of $\|u(\cdot,t)\|_\infty=O(t^{-\frac{1}{p-1}})$ arises in accordance with the "threshold modulus" so that this behavior is an exceptional case, see.[223] In this threshold modulus case, the rescaled solution $v=v(\cdot,s)$ converges to the stationary solution to (2.255), that is the self-similar solution, as $s\uparrow+\infty$, see.[166] What happens to $p=\frac{n+2}{n-2}$, $n\geq 3$ is studied by.[150]
We have

$$Lv\equiv-\Delta v-\frac{y}{2}\cdot\nabla v=-\frac{1}{K}\nabla\cdot(K\nabla v)$$

for $K = K(y) = \exp\left(|y|^2/4\right)$, and this L is realized as a self-adjoint operator in $L^2(K) = L^2(\mathbf{R}^n, K(y)dy)$ with the domain

$$H^2(K) = \left\{ v \in L^2(K) \mid D^\alpha v \in L^2(K), \ |\alpha| \le 2 \right\}.$$

Its first eigenvalue is $n/2$. Using this value and Kaplan's argument of taking the L^2 inner product between the rescaled solution $v = v(\cdot, s)$ and the first eigenfunction $\varphi_1 = \varphi_1(y) > 0$ of L normalized by $\|\varphi_1\|_1 = 1$, we can provide an alternative proof of the result[105] described above; any non-trivial solution to (2.254) blows-up in finite time if $1 < p \le p_f$, see.[85,165] The exponent $p_s = \frac{n+2}{n-2}$, $n \ge 3$ is called the *Sobolev exponent*. The non-trivial stationary solution to the prescaled equation, that is

$$-\Delta u = u^p, \ \ u > 0 \qquad \text{in } \mathbf{R}^n \tag{2.256}$$

exists if and only if $p \ge p_s$, see also (2.19). It $p_f < p < p_s$, there is a unique positive self-similar solution, that is the stationary solution to (2.255), with appropriate decay at $x = \infty$, while there is no positive self-similar solution in the case of $p \ge p_s$. Several other critical expontes have been recognized in connection with the blowup profile, blowup rate, and complete blowup of the solution, see[97] and the references therein.

Backward Self-Similar Transformation

There are vast references concerning the blowup mechanism of the solution to

$$\begin{aligned} &u_t - \Delta u = u^p, \ \ u > 0 \quad \text{in } \Omega \times (0,T) \\ &u = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{2.257}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. The principal difference between (2.252) is the appearance of the Poincaré inequality,

$$\lambda_1 \|u\|_2^2 \le \|\nabla u\|_2^2,$$

where $\lambda_1 = \lambda_1(\Omega) > 0$ denotes the first eigenvalue of $-\Delta_D$:

$$-\Delta\varphi_1 = \lambda_1 \varphi_1, \ \ \varphi_1 > 0 \quad \text{in } \Omega, \qquad \varphi_1 = 0 \quad \text{on } \partial\Omega.$$

The dominance of the L^2-norm in the global in time behavior of the solution, actually, arises with the variational structure formulated by

$$\begin{aligned} &\frac{d}{dt} J(u) = -\|u_t\|_2^2 \\ &\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u), \end{aligned}$$

where

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ I(u) &= \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}. \end{aligned}$$

A remarkable implication of this fact is

$$T = +\infty \quad \Rightarrow \quad \sup_{t \geq 0} \|u(\cdot,t)\|_2 < +\infty$$

for any $p > 1$, see.[45,245]
The comparison principle is also useful, which results in the properties of the (strongly) order preserving and the parabolic Liouville property or the intersection comparison principle.[109]
The scaling invariance (2.163) now induces the *backward self-similar transformation* defined by

$$\begin{aligned} v(y,s) &= (T-t)^{\frac{1}{p-1}} u(x,t) \\ y &= (x-x_0)/(T-t)^{1/2} \\ s &= -\log(T-t), \end{aligned}$$

where $T > 0$ is the blowup time, that is

$$\lim_{t \uparrow T} \|u(\cdot,t)\|_\infty = +\infty,$$

and $x_0 \in \overline{\Omega}$ is a blowup point. This relation implies

$$\begin{aligned} & v_s - \Delta v + \frac{y}{2} \cdot \nabla v + \frac{v}{p-1} = v^p, \ v > 0 \\ & \qquad\qquad\qquad \text{in } \textstyle\bigcup_{s > -\log T} e^{s/2} (\Omega - \{x_0\}) \times \{s\} \\ & v = 0 \qquad\qquad \text{on } \textstyle\bigcup_{s > -\log T} e^{s/2} (\partial\Omega - \{x_0\}) \times \{s\} \end{aligned}$$

and the limiting equations are

$$\begin{aligned} & v_s - \Delta v + \frac{y}{2} \cdot \nabla v + \frac{v}{p-1} = v^p \\ & v > 0 \qquad\qquad \text{in } \mathbf{R}^n \times (-\infty, +\infty) \end{aligned} \tag{2.258}$$

and

$$\begin{aligned} & v_s - \Delta v + \frac{y}{2} \cdot \nabla v + \frac{v}{p-1} = v^p \\ & v > 0 \qquad\qquad \text{in } \mathbf{R}^n_+ \times (-\infty, +\infty) \\ & v = 0 \qquad\qquad \text{on } \partial\mathbf{R}^n_+ \times (-\infty, +\infty) \end{aligned}$$

if $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively. Only the constant $v = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$ is admitted as a bounded stationary solution to (2.258) if $1 < p \leq \frac{n+2}{n-2}$, see.[120]
If $1 < p < \frac{n+2}{n-2}$, furthermore, then the global blowup rate is of type (I), that is

$$\limsup_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty < +\infty,$$

and $v(\cdot,s)$ converges to $\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$, the constant solution to (2.258), locally uniformly in $\mathbf{R}^n$ as $s \uparrow +\infty$, see.[120–122] In particular, any blowup point is of type (I), and the aggregation

dominates the concentration in the parabolic envelope, where aggregation and concentration indicate the growth of local L^1 norm and that of local L^∞ norm, respectively.
If $\Omega \subset \mathbf{R}^n$, $n \geq 3$ is convex, $p = \frac{n+2}{n-2}$, $T = T_{\max} < +\infty$, and

$$\lim_{t \uparrow T} J(u(\cdot,t)) > -\infty, \tag{2.259}$$

then the blowup rate is of type (II),

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty = +\infty.$$

The leading blowup profile, furthermore, is the normalized entire stationary solution (2.256), $u_* = u_*(x) > 0$, $u_*(0) = 1$ involved by a scaling transfomatioin, and the formation of the second collapse arises at the wedge of the principal parabolic envelope.[307] We obtain, thus, the collision of collapses in this case; compare such profiles to those of the two-dimensional Smoluchowski-Poisson equation (1.61). We emphasize that actual existence of the solution satisfying (2.259) with $T = T_{\max} < +\infty$ has not been confirmed yet, and in this connection, it should be noted that in the sub-critical case $1 < p < \frac{n+2}{n-2}$, it holds always that $T < +\infty$ if and only if

$$\lim_{t \uparrow T} J(u(\cdot,t)) = -\infty,$$

see.[119,151]
If $\Omega \subset \mathbf{R}^n$, $n \geq 3$ is convex and $p = \frac{n+2}{n-2}$, furthermore, then there is no stationary solution, and there arises the energy quantization

$$T = +\infty \quad \Rightarrow \quad \lim_{t \uparrow T} J(u(\cdot,t)) = md$$

with $m = 1, 2, \cdots$, where $d = J(u_*)$ and $u_* = u_*(x) > 0$ is the above described normalized entire stationary solution to (2.256), see.[149,152] We remind again that although (2.256) has a family of solutions in the critical case $p = \frac{n+2}{n-2}$, the energy value $d = J(u_*)$ is the same because of the scaling invariance of the problem. This property is actually the origin of the energy quantization to (2.257) with $p = \frac{n+2}{n-2}$.

2.4.2. Method of Duality

The following two paragraphs are concerned with the higher-dimensional concentration to the stationary state. Here, we provide preliminary considerations on the method of duality. This method has played a fundamental role in the study of the chemotaxis system, see §1.1, based on the symmetry of the Green's function (1.17) resulting in the weak formulation (1.33).

Pohozaev Identity

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $v = v(x)$ be a smooth solution to

$$-\Delta v = f(v) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2.260}$$

where $f = f(s)$ is a continuous function of $s \in \mathbf{R}$. In this case it holds that

$$\int_\Omega nF(v) + \frac{2-n}{2} f(v)v\, dx = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2 (x\cdot\nu), \tag{2.261}$$

where

$$F(s) = \int_0^s f(s')ds'.$$

Equality (2.261) is called the Pohozaev identity[256] and has a variety of applications.[303] Regarding v and $u = f(v)$ as the field density and the particle distribution, respectively, we obtain

$$\begin{aligned} v(x) &= \int_\Omega G(x,x')u(x')dx' \\ u\nabla v &= \nabla F(v) \end{aligned}$$

and hence

$$\nabla\cdot(\nabla F(v) - u\nabla v) = 0, \tag{2.262}$$

where $G = G(x,x')$ denotes the Green's function:

$$-\Delta_x G(\cdot,x') = \delta_{x'} \quad \text{in } \Omega, \qquad G(\cdot,x') = 0 \quad \text{on } \partial\Omega.$$

Given a C^2-function $\psi = \psi(x)$ of $x \in \overline{\Omega}$, we have

$$\begin{aligned} 0 &= \int_\Omega \nabla\cdot(\nabla F(v) - u\nabla v)\psi \\ &= \int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}F(v) - u\frac{\partial v}{\partial\nu}\right)\psi - \int_{\partial\Omega}F(v)\frac{\partial\psi}{\partial\nu} \\ &\quad + \int_\Omega F(v)\Delta\psi + u\nabla v\cdot\nabla\psi\, dx. \end{aligned}$$

Here, the boundary integrals of the right-hand side vanish by

$$\begin{aligned} F(0) &= 0 \\ F'(v) &= f(v) = u \end{aligned}$$

so that

$$\int_\Omega F(v)\Delta v + \frac{1}{2}\iint_{\Omega\times\Omega}\rho_\psi(x,x')u\otimes u = 0 \tag{2.263}$$

by the method of symmetrization, where

$$\rho_\psi(x,x') = \nabla\psi(x)\cdot\nabla_x G(x,x') + \nabla\psi(x')\cdot\nabla_{x'}G(x,x').$$

Since

$$\int_\Omega \nabla\psi(x)\cdot\nabla_{x'}G(x,x')(-\Delta v(x'))dx' = \nabla\psi\cdot\nabla v(x),$$

it holds that

$$\begin{aligned}
&\frac{1}{2}\iint_{\Omega\times\Omega}\rho_\psi(x,x')\Delta v(x)\Delta v(x')dxdx' \\
&\quad = \iint_{\Omega\times\Omega}\nabla\psi(x)\cdot\nabla_x G(x,x')(-\Delta v(x'))(-\Delta v(x))dxdx' \\
&\quad = (\nabla\psi\cdot\nabla v, -\Delta v) \\
&\quad = -\int_{\partial\Omega}\nabla\psi\cdot\nabla v\frac{\partial v}{\partial \nu} + \int_\Omega \nabla(\nabla\psi\cdot\nabla v)\cdot\nabla v \\
&\quad = -\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2\frac{\partial \psi}{\partial \nu} + \sum_{i,j}\int_\Omega \psi_{ij}v_iv_j + \sum_{i,j}\int_\Omega \psi_i v_{ij} v_j,
\end{aligned}$$

where $w_i = \frac{\partial w}{\partial x_i}$, $w_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}$, and so forth.
Using

$$\begin{aligned}
I_{ij} &= \int_\Omega \psi_i v_{ij} v_j \\
&= \int_{\partial\Omega}\psi_i \nu_i v_j^2 - \int_\Omega v_j(\psi_i v_j)_i \\
&= \int_{\partial\Omega}\psi_i \nu_i v_j^2 - \int_\Omega \psi_{ii} v_j^2 - I_{ij},
\end{aligned}$$

we have

$$\sum_{i,j}\int_\Omega \psi_i v_{ij} v_j = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2\frac{\partial \psi}{\partial \nu} - \frac{1}{2}\int_\Omega \Delta\psi|\nabla v|^2$$

and, therefore,

$$\begin{aligned}
&\frac{1}{2}\iint_{\Omega\times\Omega}\rho_\psi(x,x')\Delta v(x)\Delta v(x')dxdx' \\
&= -\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2\frac{\partial \psi}{\partial \nu} - \frac{1}{2}\int_\Omega \Delta\psi|\nabla v|^2 + \sum_{i,j}\int_\Omega \psi_{ij}v_iv_j. \qquad (2.264)
\end{aligned}$$

We obtain

$$\int_\Omega F(v)\Delta\psi + \sum_{i,j}\int_\Omega \psi_{ij}v_iv_j = \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\right)^2\frac{\partial \psi}{\partial \nu} + \frac{1}{2}\int_\Omega \Delta\psi|\nabla v|^2 \qquad (2.265)$$

by (2.263)-(2.264), and hence (2.261), putting $\psi = |x|^2$. □

Concentration Mass Estimate

We can show that (2.220) implies (2.223) by the method of symmetrization. For this purpose, we apply (2.222) and confirm the well-definedness of

$$w(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\log\frac{1}{|x-x'|}\cdot e^{v(x')}dx' \qquad (2.266)$$

with the estimate

$$|w(x)| \leq C(1+\log(1+|x|)). \tag{2.267}$$

Then, Liouville's theorem guarantees, see,[259]

$$v = w + \text{constant}$$

by $\Delta(v-w)=0$, (2.222), and (2.267). Writing

$$u = e^v, \tag{2.268}$$

we have

$$\nabla u = u\nabla v = u\nabla w$$

and hence

$$\nabla\cdot(\nabla u - u\nabla w) = 0 \qquad \text{in } \mathbf{R}^2. \tag{2.269}$$

Equations (2.266), (2.268), and (2.269) comprise of the stationary system of chemotaxis, and therefore, we obtain

$$\int_{\mathbf{R}^2} \Delta\varphi(x)\cdot u(x)dx + \frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2} \rho_\varphi^0(x,x')u(x)u(x')dxdx' = 0 \tag{2.270}$$

for $\varphi \in C_0^2(\mathbf{R}^2)$, where

$$\rho_\varphi^0(x,x') = -\frac{1}{2\pi}\cdot\frac{(\nabla\varphi(x)-\nabla\varphi(x'))\cdot(x-x')}{|x-x'|^2}.$$

Using $c=c(s)$ of (1.46), we have $\delta>0$ determined by M such that

$$\int_{\mathbf{R}^2}\left(c(|x|^2)+1\right)u(x)dx \geq \delta$$

in case

$$\int_{\mathbf{R}^2} e^v = \int_{\mathbf{R}^2} u = M > 8\pi$$

similarly to[174] or §1.1.6. This time, $|y|^2/4$ factor of the right-hand side of (1.42) does not appear, and we obtain the scaling

$$u_\mu(x) = \mu^2 u(\mu x), \qquad v_\mu(x) = v(\mu x) + 2\log\mu.$$

Then it follows that

$$M = \int_{\mathbf{R}^2} u = \int_{\mathbf{R}^2} u_\mu$$
$$\int_{\mathbf{R}^2}\left(c(|x|^2)+1\right)u_\mu(x)dx = \int_{\mathbf{R}^2}\left(c(\mu^{-1}|x|^2)+1\right)u(x)dx$$

and therefore, we can find $\mu \gg 1$ satisfying

$$\int_{\mathbf{R}^2}\left(c(|x|^2)+1\right)u_\mu(x)dx < \delta,$$

a contradiction. □

Two-Dimensional Mass Quantization Revisited

Putting

$$w = v + \log \lambda - \log \int_{\Omega} e^{v}$$

in (2.155), we obtain

$$-\Delta w = e^{w} \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma = \partial\Omega$$
$$\int_{\Omega} e^{w} = \lambda. \tag{2.271}$$

If $w = w(x)$ solves (2.271), conversely, then $v = w - w_{\Gamma}$ is a solution to (2.155). Theorem 2.7 then implies the quantized blowup mechanism to (2.271).

Theorem 2.28 ([308]). *Assume that $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\{(\lambda_k, w^k)\}$ is a solution sequence to (2.271) satisfying $\lambda_k \to \lambda_0$. Then passing to a subsequence the following alternatives hold:*

(1) $\|w^k\|_{\infty} = O(1)$.
(2) $\sup_{\Omega} w^k \to -\infty$.
(3) $\lambda_0 = 8\pi\ell$ *for some $\ell \in \mathbf{N}$, and there exist $x_j^* \in \Omega$, $j = 1, \ldots, \ell$, satisfying (2.158) and $x_k^j \to x_j^*$, such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \{x_1^*, \ldots, x_\ell^*\}$, and*

$$e^{w^k} dx \rightharpoonup \sum_j 8\pi \delta_{x_j^*}(dx) \qquad \text{in } \mathscr{M}(\overline{\Omega}).$$

Thus $\mathscr{S} = \{x_1^*, \ldots, x_\ell^*\}$ is the blowup set of $\{w^k\}$.

If we take $u = e^{w}$ in (2.271), then it follows that

$$\nabla u = u \nabla w$$
$$w - w_{\Gamma} = \int_{\Omega} G(\cdot, x') u(x') dx',$$

where $G = G(x,x')$ denotes the Green's function for $-\Delta$ provided with the boundary condition $\cdot|_{\partial\Omega} = 0$. This relation implies

$$\int_{\Omega} u \nabla \cdot \psi + \iint_{\Omega \times \Omega} \psi(x) \cdot \nabla_x G(x,x') u(x) u(x') dx dx' = 0 \tag{2.272}$$

for any $\psi \in C_0^2(\Omega)^2$.
Equality (2.272) is the dual weak formulation of (2.271). Using this formula, we can show that if $\{(w^k, \lambda_k)\}$ is a solution sequence to (2.271) satisfying

$$e^{w^k} dx \rightharpoonup \sum_{x_0 \in \mathscr{S}} m(x_0) \delta_{x_0}(dx) \qquad \text{in } \mathscr{M}(\overline{\Omega})$$

with $m(x_0) > 0$, then it holds that $m(x_0) = 8\pi$ and (2.158). The proof is similar to.[234] Mass quantization, conversely, implies the residual vanishing, see[235,304] for details.

2.4.3. Higher-Dimensional Quantization

Problem (2.271) is regarded as a free boundary problem associated with the plasma confinement, where $\{w > 0\}$ indicates the plasma region.[103] The higher-dimensional mass quantization is observed in an analogous free boundary problem

$$-\Delta w = w_+^q \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma$$
$$\int_\Omega w_+^q = \lambda, \tag{2.273}$$

where $\Omega \subset \mathbf{R}^n$, $n \geq 3$ is a bounded domain with smooth boundary $\partial\Omega = \Gamma$, and $q = \frac{n}{n-2}$. This problem describes the stationary state of the degenerate parabolic equation derived from the kinetic theory, see §2.2.6. It is also the equilibrium self-gravitating fluid equation described by the field component, and the problem of plasma confinement, see §2.1.3.
First, we obtain $v_k = w^k - w_\Gamma^k \geq 0$ in Ω by the maximum principle similarly to the two-dimensional case. Here, the quantized value $m_* > 0$ is defined by

$$m_* = \int_B U^q$$

for $U = U(x)$ satisfying

$$-\Delta U = U^q, \ \ U > 0 \quad \text{in } B, \qquad U = 0 \quad \text{on } \partial B$$

with $B = B(0,R)$. As is examined in §2.1.2, this $U = U(x)$ is radially symmetric and exists uniquely for each $R > 0$, while m_* is independent of $R > 0$. In the following theorem, $G = G(x,x')$ denotes the Green's function of $-\Delta$ on Ω with the Dirichlet boundary condition and

$$R(x) = \left[G(x,x') - \Gamma(x-x')\right]_{x'=x},$$

where

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}}$$

is the fundamental solution to $-\Delta$ and ω_{n-1} is the $(n-1)$ dimensional volume of the boundary of the unit ball in $\mathbf{R}^n$.

Theorem 2.29 ([308]). *If $\Omega \subset \mathbf{R}^n$, $n \geq 3$ is a bounded domain with smooth boundary $\partial\Omega$ and $\left\{(\lambda_k, w^k)\right\}$ is a solution sequence to (2.273) with $q = \frac{n}{n-2}$ satisfying $\lambda_k \to \lambda_0$, then passing to a subsequence the following alternatives hold:*

(1) $\left\|w^k\right\|_\infty = O(1)$.
(2) $\sup_\Omega w^k \to -\infty$.
(3) *$\lambda_0 = m_*\ell$ for some $\ell \in \mathbf{N}$, and there exist $x_j^* \in \Omega$ and $x_k^j \to x_j^*$ $(j = 1,\ldots,\ell)$, where $\mathscr{S} = \left\{x_1^*,\ldots,x_\ell^*\right\} \subset \Omega$ coincides with the blowup set of $\left\{w^k\right\}$ on $\overline{\Omega}$ satisfying (2.158), $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \mathscr{S}$, and*

$$w^k(x)_+^q dx \rightharpoonup \sum_j m_* \delta_{x_j^*}(dx) \qquad \text{in } \mathscr{M}(\overline{\Omega}).$$

There are several results on the actual existence of the solution sequence described in the above theorem.[332,333,340]

Local version comparable to Theorems 1.23-1.24 also holds. Actually, there are ε-regularity, self-similarity, classification of the entire solution, and sup + inf inequality, and these structures guarantee the following theorem similarly to the two-dimensional case of Theorem 2.11. A slight difference to Theorem 2.28 is that the entire solution

$$-\Delta w = w_+^q, \quad w \leq w(0) = 1 \qquad \text{in } \mathbf{R}^n$$
$$\int_{\mathbf{R}^n} w_+^q < +\infty$$

provided with a compact support to $w_+^q dx$. This property, however, does not cause any difficulties.

Theorem 2.30 ([331]). *If $\Omega \subset \mathbf{R}^n$, $n \geq 3$ is a bounded domain and $w = w^k$, $k = 1, 2, \ldots$, satisfies*

$$-\Delta w = w_+^q \quad \text{in } \Omega, \qquad \int_\Omega w_+^q \leq C$$

for $q = \frac{n}{n-2}$ and $C > 0$, then passing to a subsequence, we obtain the following alternatives.

(1) $\{w^k\}$ is locally uniformly bounded in Ω.
(2) $w^k \to -\infty$ locally uniformly in Ω.
(3) There exist $\ell \in \mathbf{N}$, x_j^* $(j = 1, \ldots, \ell)$, and $x_k^j \to x_j^*$ such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\Omega \setminus \{x_1^*, \ldots, x_\ell^*\}$, and

$$w^k(x)_+^q dx \rightharpoonup \sum_j m_* n_j \delta_{x_j^*}(dx) \qquad \text{in } \mathscr{M}(\Omega),$$

where $n_j \in \mathbf{N}$.

Here, we emphasize that Theorem 2.30 is used in the proof of Theorem 2.29, and we recall that Theorem 2.7 is proven independently of Theorem 2.11.

2.4.4. Dimension Control of the Blowup Set

Even the general blowup mechanism in high-space dimensions is enclosed by smaller sets. In the final paragraph, we describe a simple case of,[309] see also[267] for the other result.

First, given $1 \leq p < n$, we put

$$K^p = \{f \in L^{p^*}(\mathbf{R}^n, \mathbf{R}) \mid \nabla f \in L^p(\mathbf{R}^n, \mathbf{R})\}$$

for $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, and define the p-capacity of a subset $A \subset \mathbf{R}^n$ by

$$\mathrm{Cap}_p(A) = \inf\left\{ \int_{\mathbf{R}^n} |\nabla f|^p \mid f \geq 0,\ f \in K^p,\ A \subset \{f \geq 1\}^\circ \right\},$$

where B° denotes the interior of $B \subset \mathbf{R}^n$. This Cap_p is an outer measure on $\mathbf{R}^n$ satisfying

$$\mathrm{Cap}_p(A) \leq CH^{n-p}(A), \qquad A \subset \mathbf{R}^n,$$

where H^s denotes the s-dimensional Hausdorff measure:

$$\begin{aligned}
H^s(A) &= \lim_{\delta \downarrow 0} H^s_\delta(A) \\
H^s_\delta(A) &= \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) r_j^s \mid A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j),\ r_j < \delta \right\} \\
\alpha(s) &= \frac{\pi^{s/2}}{\Gamma(\frac{s}{2} - 1)} \\
\Gamma(s) &= \int_0^\infty e^{-t} t^{s-1} dt.
\end{aligned}$$

We have

$$\begin{aligned}
H^{n-p}(A) < +\infty,\ 1 < p < n \quad &\Rightarrow \quad \mathrm{Cap}_p(A) = 0 \\
\mathrm{Cap}_p(A) = 0,\ s > n - p \quad &\Rightarrow \quad H^s(A) = 0,
\end{aligned}$$

see.[87]

Theorem 2.31. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$ be a bounded open set, $T > 0$, and*

$$u = u(x,t) : \Omega \times [0,T] \to (-\infty, +\infty]$$

be a continuous function satisfying

$$\begin{aligned}
D &= \bigcup_{0 \leq t \leq T} D(t) \times \{t\} \subset \Omega \times [0,T] \\
u_t - \Delta u &\geq 0 \qquad \text{in } \Omega \times (0,T) \setminus D, \qquad (2.274)
\end{aligned}$$

where

$$D(t) = \overline{\{x \in \Omega \mid u(x,t) = +\infty\}}$$

and suppose that $u = u(x,t)$ is Lipschitz continuous near $\partial\Omega$ uniformly in $t \in [0,T]$. Then, it holds that

$$\int_0^T Cap_2(D(t))dt \leq \frac{L^n(\Omega)}{2}. \qquad (2.275)$$

The second relation of (2.274) is taken in the distributional sense, so that

$$\iint_{\Omega \times (0,T) \setminus D} u\varphi_t + \iint_{\Omega \times (0,T) \setminus D} u\Delta\varphi \leq 0$$

for any $\varphi = \varphi(x) \geq 0$ in $\varphi \in C_0^\infty(\Omega \times (0,T) \setminus D)$. If u is independent of t, then we obtain

$$\mathrm{Cap}_2(D) = 0$$

by (2.275), where $D = D(t)$. If Ω is convex, on the other hand, we have the boundary estimate for solutions to the semilinear elliptic equation, see.[73,114] Then, the following theorem follows from the proof of Theorem 2.31, where $d_H(A)$ denotes the Hausdorff dimension:

$$d_H(A) = \inf\{s \geq 0 \mid H^s(A) = 0\}.$$

Theorem 2.32. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$ be a bounded convex domain, $f = f(u,\lambda)$ be smooth in $u \geq 0$, and $\{(u_k, \lambda_k)\}_k$ be a solution sequence to*

$$-\Delta u = f(u,\lambda) \geq 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

satisfying

$$\int_\Omega f(u_k, \lambda_k) dx \leq C$$
$$\|u_k\|_\infty \to +\infty.$$

Then, it holds that $Cap_2(\mathscr{S}) = 0$ and in particular,

$$d_H(\mathscr{S}) \leq n-2,$$

where

$$\mathscr{S} = \{x_0 \in \overline{\Omega} \mid \text{there exists } x_k \to x_0 \text{ such that } u_k(x_k) \to +\infty\}$$

denotes the blowup set.

We have

$$u_k \to +\infty \qquad \text{locally uniformly in } \Omega$$

in the case of

$$\int_\Omega f(u_k, \lambda_k) dx \to +\infty.$$

The proof of this fact of entire blowup is similar to the two-dimensional case, see[213] and §2.3.4.

2.4.5. Summary

Although the non-stationary blowup mechanism in higher-space dimension is a problem for future, we have evidence as well as new overviews for the mass and energy quantizations.

(1) The method of scaling reveals an even higher-dimensional blowup mechanism.
(2) For the semilinear parabolic equation with the critical Sobolev exponent, there is an energy quantization to the non-compact solution sequence with bounded total energy. If it occurs in a finite time, then we obtain the formation of sub-collapse and the collision of collapses at the wedge of the parabolic envelope.
(3) There is mass quantization for the stationary higher-dimensional state to the free boundary problem with the critical exponent.

(4) A very general dimension control of the blowup set holds in terms of the capacity.

(5) Type (II) blowup rate, formation of sub-collapses, and possible collision of collapses are observed when the free energies are bounded in the energy quantization, such as harmonic heat flow and semilinear parabolic equation with critical Sobolev exponent. In the two-dimensional Smoluchowski-Poisson equation, on the other hand, we have always type (II) blowup rate, formation of a sub-collapse, and collisionless of collapses as a consequence of the mass quantization.

References

1. R.D. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
2. N.D. Alikakos, P.W. Bates, and X. Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Rational Mech. Anal. **128** (1994) 165–205.
3. N.D. Alikakos and G. Fusco, *Slow dynamics for the Cahn-Hilliard equation in higher space dimensions: the motion of bubbles*, Arch. Rational Mech. Anal. **141** (1998) 1–61.
4. S.M. Allen and J.W. Cahn, *A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. **27** (1979) 1085–1095.
5. W. Alt, *Biased random walk models for chemotaxis and related diffusion approximations*, J. Math. Biol. **9** (1980) 147–177.
6. H.W. Alt and I. Pawlow, *Dynamics of non-isothermal phase separation*, In; *Free Boundary Value Problems* (edited by K.-H. Hoffmann and J. Sprekels), pp. 1–26, Birkhäuser-Veralag, Basel, 1990.
7. H.W. Alt and I. Pawlow, *Mathematical model of dynamics of non-isothermal phase separation*, Physica D **59** (1992) 389–416.
8. A.R.A. Anderson and M.A.J. Chaplain, *Continuous and discrete mathematical models of tumor-induced angiogenesis*, Bull. Math. Biol. **60** (1998) 857–899.
9. S. Angenent, *The Morse-Smale property for a semilinear parabolic equation*, J. Differential Equations **62** (1986) 427–442.
10. S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988) 79–96.
11. V.I. Arnold and B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer, New York, 1998.
12. A. Bahri, Ĉritical Points at Infinity in Some Variational Problems, Logman, New York, 1988.
13. C. Bandle, *Isoperimetric inequalities for a nonlinear eigenvalue problem*, Proc. Amer. Math. Soc. **56** (1976) 243–246.
14. C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, London, 1980.
15. S. Baraket and F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension* 2, Calc. Vari. in PDE **6** (1998) 1–38.
16. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Nordhoff, Leiden, 1976.
17. G. Barles, H.M. Soner, and P.E. Souganidis, *Front propagation and phase field theory*, SIAM J. Cont. Optim. **31** (1993) 439–469.
18. J. Bartz, M. Struwe, and R. Ye, *A new approach to the Ricci flow on* S^2, Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV **21** (1994) 475–482.
19. G. Bates and P.C. Fife, *Dynamics of nucleation for the Cahn-Hilliard equation*, SIAM J. Appl. Math. **53** (1993) 990–1008.
20. F. Bavaud, *Equilibrium properties of the Vlasov functional: the generalized Poisson-Boltzmann-Emden equation*, Rev. Mod. Phys. **63** (1991) 129–149.
21. H. Berestycki and H. Brezis, *On a free boundary problem arising in plasma physics*, Nonlinear Analysis **4** (1980) 415–436.
22. F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, Boston, 1994.
23. M. Bezard, *Existence locale de solutions pour les équation d'Euler-Poisson*, Japan J. Indust. Appl. Math. **10** (1993) 431–450.
24. P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998) 715–743.
25. P. Biler, D. Hilhorst, and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, II*, Colloq. Math. **67** (1994) 297–308.

T. Suzuki, *Mean Field Theories and Dual Variation*, Atlantis Studies in Mathematics for Engineering and Science 2, DOI 10.1007/978-94-91216-22-0,

26. P. Biler and R. Stańczy, *Mean field models for self-gravitating particles*, Folia Matematica **13** (2006) 3–19.
27. E.B. Bogomol'nyi, *The stability of classical solutions*, Sov. J. Nucl. Phys. **24** (1976) 449–454.
28. F. Bouchut, *Existence and uniqueness of global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions*, J. Funct. Anal. **111** (1993) 239–258.
29. F. Bouchut, F. Golse, and M. Pulvirenti, *Kinetic Equations and Asymptotic Theory*, Gauthier-Villars, Paris, 2000.
30. H. Brezis, *Analyse Fonctionnelle, Théorie et Applications*, Masson, Paris, 1983.
31. H. Brezis, Y.Y. Li, and I. Shafrir, *A* $\sup + \inf$ *inequality for some nonlinear elliptic equations involving exponential nonlinearities*, J. Funct. Anal. **115** (1993) 344–358.
32. H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations **16** (1991) 1223–1253.
33. H. Brezis and W. Strauss, *Semi-linear second-order elliptic equations in* L^1, J. Math. Soc. Japan **25** (1973) 565–590.
34. M. Brokate and J. Sprekels, *Hysteresis and Phased Transitions*, Springer-Verlag, New York, 1996.
35. L. Bronsard and R.V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991) 211–237.
36. J.P. Burelbach, S.G. Bankoff, and S.H. Davis, *Nonlinear stability of evaporating/condensing liquid films*, J. Fluid. Mech., **195** (1988) 463–494.
37. L. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989) 271–297.
38. G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1993) 9900–1008.
39. E. Caglioti, P.-L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Comm. Math. Phys. **143** (1992) 501–525.
40. E. Caglioti, P.-L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Part II*, Comm. Math. Phys. **174** (1995) 229–260.
41. J.W. Cahn, *Phase separation by spinodal decomposition in isotropic systems*, J. Chem. Phys. **42** (1965) 93–99.
42. J.W. Cahn and J.E. Hilliard, *Free energy of a nonuniform system I. Interfacial free energy*, J. Chem. Phys. **28** (1958) 258–267.
43. E. Cancés, M. Defranceschi, W. Kutzelnigg, C. Le Bris, and Y. Maday, *Computational Quantum Chemistry: A Primer*, In; *Computational Chemistry* (edited by C. Le Bris), pp. 3–270, Elsevier, Amsterdam, 2003.
44. R.C. Casten and C.J. Holland, *Instability results for reaction diffusion equations with Neumann boundary condition*, J. Differential Equations **27** (1978) 266–273,
45. T. Cazenave and P.-L. Lions, *Solutions globales déquations de la chaleur semi lineaires*, Comm. Partial Differential Equations **9** (1984) 955–978.
46. S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Univ. Chicago Press, 1939.
47. S.-Y.A. Chang, C.-C. Chen, and C.-S. Lin, *Extremal functions for a mean field equations in two dimension*, In; Lectures on Partial Differential Equations (ed. S.-Y.A. Chang, C.-S. Lin, and S.-T. Yau), International Press, New York, 2003.
48. S.-Y.A. Chang and P.-C. Yang, *Prescribing Gaussian curvature on* S^2, Acta Math. **159** (1987) 215–259.
49. S.-Y.A. Chang and P.-C. Yang, *Conformal deformation of metrics on* S^2, J. Differential Geometry **27** (1988) 259–296.
50. S. Chanillo and M. Kiessling, *Rotational symmetry of solutions of some nonlinear problems in*

statistical mechanics and in geometry, Comm. Math. Phys. **160** (1994) 217–238.
51. P.-H. Chavanis, *Generalized kinetic equations and effective thermodynamics*, Banach Center Publications **66** (2004) 79–101.
52. P.-H. Chavanis and C. Sire, *Anormalous diffusion and collapse of self-gravitating Langevin particles in D dimensions*, Phys. Rev. E **69** (2004) 016116.
53. C.-C. Chen and C.-S. Lin, *A sharp* $\sup + \inf$ *inequality for a semilinear elliptic equation in* $\mathbf{R}^2$, Comm. Anal. Geom. **6** (1998) 1–19.
54. C.-C. Chen and C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55** (2002) 728–771.
55. C.-C. Chen and C.-S. Lin, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56** (2003) 1667–1727.
56. W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991) 615–622.
57. X. Chen, *Remarks on the existence of branch bubbles on the blowup analysis of equation* $-\Delta u = e^{2u}$ *in dimension two*, Comm. Anal. Geom. **7** (1999) 295–302.
58. X.-Y. Chen, *Generation and propagation of the interface for reaction-diffusion equation*, J. Differential Equations **96** (1992) 116–141.
59. Y.-G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry **33** (1991) 744–486.
60. K.-S. Cheng and C.-S. Lin, *On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in* $\mathbf{R}^2$, Math. Ann. **308** (1997) 119–139.
61. S. Childress and J.K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci. **56** (1981) 217–237.
62. M. Chipot, I. Shafrir, and G. Wolansky, *On the solutions of Liouville systems*, J. Differential Equations **140** (1997) 59–105.
63. B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Amer. Math. Soc. Providence, 2004.
64. R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993) 247–286.
65. J.B. Collins and H. Levine, *Diffuse interface model of diffusion-limited crystal growth*, Phys. Rev. B **31** (1985) 6119–6122, **33** (1985) 2020.
66. L. Corrias, B. Perthame, and H. Zaag, *A chemotaxis model motivated by angiogenesis*, C. R. Acad. Sci. Paris, Ser. I **336** (2003) 141–146.
67. L. Corrias, B. Perthame, and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math. **72** (2004) 1–28.
68. M.G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992) 1–67.
69. M.G. Crandall and T. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–293.
70. G. Dal Maso, *An Introduction to* Γ*-Convergence*, Birkhäuser, Boston, 1993.
71. A. Damlamian, *Application de la dualité non convexe à un problème non linéaire à frontière libre*, C. R. Acad. Sci. Paris, **286A** (1978) 153–155.
72. E.N. Dancer, *On stability and Hopf bifurcations for chemotaxis systems*, Meth. Appl. Anal. **8** (2001) 245–256.
73. D.G. De Figueiredo, P.-L. Lions, and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. **61** (1982) 41–63.
74. P.G. de Gennes, *Wetting: statics and dynamics*, Rev. Modern Phys. **57** (1985) 827–863.
75. S.R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics*, Dover, New York, 1984.
76. J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc. **4** (1991) 553–586.
77. P. DeMottoni and M. Schatzman, *Development of interfaces in* $\mathbf{R}^N$, Proc. Roy. Soc. Edinb. **116A** (1990) 207–220.

78. Y. Deng, T.-P. Liu, Y. Yang, and Z.-a. Yao, *Solutions of Euler-Poisson equations for gaseous stars*, Arch. Rational Mech. Anal. **164** (2002) 261–285.
79. W. Ding, and G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom. **4** (1995) 543–554.
80. J.L. Doob, *Stochastic Processes*, John Wiley and Sons, Inc., New York, 1953.
81. B. Ducomet, E. Feireisl, H. Petzeltová, and I. Straskraba, *Global in time weak solutions for compressible barotropic self-gravitating fluids*, Discrete and Continuous Dynamical Systems **11** (2004) 113–130.
82. T. Eguchi, K. Oki, and S. Matsumura, *Kinetics of ordering with phase separation*, Math. Res. Soc. Symp. Proc. **21**, pp. 589–594, Elsevier, 1984.
83. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
84. G.L. Eryink and H. Spohn, *Negative-temperature states and large scale, long-visited vortices in two-dimensional turbulence*, J. Stat. Phys. **70** (1993) 833–886.
85. M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions for the heat equation*, Nonlinear Analysis **33** (1987) 51–69.
86. L.C. Evans, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J. **42** (1993) 533–557.
87. L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
88. L.C. Evans, H.M. Soner, and P.E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992) 1097–1123.
89. L.C. Evans and J. Spruck, *Motion of level sets by mean curvature, I.*, J. Differential Geometry **33** (1991) 635–681.
90. L.C. Evans and J. Spruck, *Motion of level sets by mean curvature, II.*, Trans. Amer. Math. Soc. **330** (1992) 321–332.
91. F. Falk, *Ginzburg-Landau theory of static domain walls in shape-memory alloys*, Physica B **51** (1983) 177–185.
92. F. Falk, *Ginzburg-Landau theory and solitary waves in shape-memory alloys*, Physica B **54** (1984) 159–167.
93. F. Falk and P. Konopka, *Three-dimensional Landau theory describing the martensitic phase transition of shape-memory alloys*, J. Phys.: Condens. Matter **2** (1990) 61–77.
94. E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2004.
95. B. Fiedler and C. Rocha, *Heteroclinic orbits of semilinear parabolic equations*, J. Differential Equations **125** (1996) 239–281.
96. P.C. Fife, *Models for phase separation and their mathematics*, Elec. J. Differential Equations **2000** (2000) no. 48, 1–26.
97. M. Fila, *Blow-up of solutions of supercritical parabolic equations*, In; Handbook of Differential Equations, Evolutionary Equations (ed. C.M. Dafermos and E. Feireisl), vol. 2, pp. 105–158, Elsevier, Amsterdam. 2005.
98. R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophys. J. **1** (1961) 445–466.
99. G. Fix, *Phase field method for free boundary problems*, In; *Free Boundary Problems*, (ed. A. Fasano and M. Primicerio), Pitmann, London, 1983.
100. G.B. Folland, *Introduction to Partial Differential Equations*, second edition, Princeton University Press, Princeton, 1995.
101. L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helvetici **68** (1993) 415–454.
102. M.A. Fontelos, A. Friedman, and B. Hu, *Mathematical analysis of a model for the initiation of angiogenesis*, SIAM J. Math. Anal. **33** (2002) 1330–1355.

103. A. Friedman, *Variational Principles and Free-Boundary Problems*, John Wiley and Sons, New York, 1982.
104. A. Friedman and J.I. Tello, *Stability of solutions of chemotaxis equations in reinforced random walks*, J. Math. Anal. Appl. **272** (2002) 138–163.
105. H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sec. IA **13** (1966) 109–124.
106. H. Fujita, N. Saito, and T. Suzuki, *Operator Theory and Numerical Methods*, North-Holland, Amsterdam, 2001.
107. G. Fusco and C. Rocha, *A permutation related to the dynamics of a scalar parabolic PDE*, J. Differential Equations **91** (1991) 111–137.
108. H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. **195** (1998) 77–114.
109. V.A. Galaktionov, *Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications*, Chapman and Hall/CRC, Boca Raton, 2004.
110. I. Gallagher and T. Gallay, *Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity*, Math. Ann. **332** (2005) 287–327.
111. T. Gallay and C.E. Wayne, *Global stability of vortex solutions of the two-dimensional Navier-Stokes equation*, Comm. Math. Phys. **255** (2005) 97–129.
112. P. Gamblin, *Solution regulière a temps petits pour l'equations d'Euler-Poisson*, Comm. Partial Differential Equatioins **18** (1993) 731–745.
113. D.Y. Gao, *Duality Principle in Nonconvex Systems, Theory, Methods, and Applications*, Kluwer Academic Publ. Dordrecht, 2000.
114. B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979) 209–243.
115. B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Adv. Math. Suppl. Stud. **7A** (1981) 369–402.
116. B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981) 525–598.
117. B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Partial Differential Equations **6** (1981) 883–901.
118. A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik **12** (1972) 30–39.
119. Y. Giga, *A bound for global solutions of semilinear heat equations*, Comm. Math. Phys. **103** (1986) 415–421.
120. Y. Giga and R.V. Kohn, *Asymptotic self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985) 297–319.
121. Y. Giga and R.V. Kohn, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987) 1–40.
122. Y. Giga and R.V. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989) 845–884.
123. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
124. V.L. Ginzburg and L.D. Landau, *On the theory of superconductivity*, In; *Collected Papers of L.D. Landau* (ed. D. Ter Harr), pp. 546–568, Pergamon, New York, 1965.
125. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.
126. C. Gui, W.-M. Ni, and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation*, Comm. Pure Appl. Math. **45** (1992) 1153–1181.
127. Z. Guo, *On the symmetry of positive solutions of the Lane-Emden equation with supercritical exponent*, Adv. Differential Equations **7** (2002) 641–666.
128. J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI,

1988.
129. J.K. Hale and G. Raugel, *Convergence in gradient-like systems with applications to PDE*, Z. Angew. Math. Phys. **43** (1992) 63–124.
130. B.I. Halperin, P.C. Hohenberg, and S.-k. Ma, *Renormalization-group methods for critical dynamics: I, Recursion relations and effects of energy conservation*, Phys. Rev. B **10** (1974) 139–153.
131. R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306.
132. R.S. Hamilton, *The Ricci flow on surfaces*, Contem. Math. **71**, Amer. Math. Soc. Providence, pp. 237–262, 1988.
133. T. Hanada, N. Ishimura, and M. Nakamura, *On the Eguchi-Oki-Matsumura equation for phase separation in one space dimension*, SIAM J. Math. Anal. **36** (2004) 463–478.
134. G. Harada, T. Nagai, T. Senba, and T. Suzuki, *Concentration lemma, Brezis-Merle type inequality, and a parabolic system of chemotaxis*, Adv. Differential Equations **6** (2001) 1255–1280.
135. A. Harrabi and S. Rebhi, *Solutions for superlinear elliptic equations and their Morse indices, I.*, Duke Math. J. **94** (1998) 141–157.
136. A. Harrabi and S. Rebhi, *Solutions for superlinear elliptic equations and their Morse indices, II.*, Duke Math. J. **94** (1998) 159–179.
137. K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic equations*, Proc. Japan Acad. Ser. A **49** (1973) 503–505.
138. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840**, Springer-Verlag, Berlin, 1981.
139. D. Henry, *Some infinite dimensional Morse-Smale systems defined by parabolic differential equations*, J. Differential Equations **59** (1985) 165–205.
140. M.A. Herrero, E. Medina, and J.J.L. Velázquez, *Self-similar blow-up for a reaction-diffusion system*, J. Comp. Anal. Math. **97** (1998) 99–119.
141. M.A. Herrero, E. Medina, and J.J.L. Velázquez, *Finite-time aggregation into a single point in a reaction-diffusion system*, Nonlinearity **10** (1997) 1739–1754.
142. M.A. Herrero and J.J.L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann. **306** (1996) 583–623.
143. E. Hewitt and L.J. Savage, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc. **80** (1955) 470–501.
144. T. Hillen and H.G. Othmer, *The diffusion limit of transport equations derived from velocity jump processes*, SIAM J. Appl. Math. **61** (2000) 751–775.
145. W. Hirsh, *Stability and convergence in strongly monotone dynamical systems*, J. Rine Angew. Math. **383** (1988) 1–53.
146. P.C. Hohenberg and B.I. Halperin, *Theory of dynamic critical phenomena*, Rev. Mod. Phys. **49** (1977) 435–479.
147. C.W. Hong, *A best constant and the Gaussian curvature*, Proc. Amer. Math. Soc. **97** (1987) 737–747.
148. S.-Z. Huang, *Gradient Inequalities*, Amer. Math. Soc. Providence, 2006.
149. R. Ikehata, *The Palais-Smale condition for the energy of some semilinear parabolic equations*, Hiroshima Math. J. **30** (2000) 117–127.
150. R. Ikehata, M. Ishiwata, and T. Suzuki, *Semilinear parabolic equation in* $\mathbf{R}^N$ *with critical Sobolev exponent*, preprint.
151. R. Ikehata and T. Suzuki, *Stable and unstable sets for evolution equations of parabolic and hyperbolic type*, Hiroshima Math. J. **26** (1996) 475–491.
152. R. Ikehata and T. Suzuki, *Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions*, Differential and Integral Equations **13** (2000) 437–477.

153. T. Ilmannen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geometry **38** (1993) 417–461.
154. A. Ito and T. Suzuki, *Asymptotic behavior of the solution to the non-isothermal phase field equation*, Nonlinear Analysis **64** (2006) 2454–2479.
155. A. Ito and T. Suzuki, *Asymptotic behavior of the solution to the non-isothermal phase separation*, Nonlinear Analysis **68** (2008) 1825–1843.
156. T. Iwaniec and A. Verde, *On the operator* $\mathscr{L}(f) = f \log|f|$, J. Funct. Anal. **169** (1999) 391–420.
157. W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992) 819–824.
158. J. Jost, *Two-Dimensional Geometric Variational Problems*, John Wiley and Sons, Chichester, 1991.
159. G. Joyce and D. Montgomery, *Negative temperature states for two-dimensional guiding-centre plasma*, J. Plasma Phys. **10** (1973) 107–121.
160. S.L. Kamenomostskaja, *On the Stefan probem* (in Russian), Mat. Sbornik **53** (1961) 488–514.
161. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.
162. T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems*, Accademia Nazionale Dei Lincei, Scuola Normale Superiore, Pisa, 1985.
163. N. Kavallaris and T. Suzuki, *On the finite-time blowup of a parabolic equation describing chemotaxis*, Differential and Integral Equations **20** (2007) 293–308.
164. N. Kavallaris and T. Suzuki, *An analytic approach to the normalized Ricci flow-like equation*, preprint.
165. O. Kavian, *Remarks on the large time behavior of a nonlinear diffusion equation*, Ann. Inst. Henri Poincaré, Analyse non linéaire **4** (1987) 423–452.
166. T. Kawanago, *Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity*, Ann. Inst. Henri Poincaré, Analyse non linèaire **13** (1996) 1–15.
167. J.L. Kazdan and F.W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975) 567–597.
168. J.P. Keener, *Activator and inhibitors in pattern formation*, Stud. Appl. Math. **59** (1978) 580–586.
169. E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970) 399–415.
170. M.K.H. Kiessling, *Statistical mechanics of classical particles with logarithmic interaction*, Comm. Pure Appl. Math. **46** (1993) 27–56.
171. K. Kobayashi, T. Sirao, and H. Tanaka, *On the blowing up problem for semiliner heat equations*, J. Math. Soc. Japan **29** (1977) 407–424.
172. T. Kobayashi and T. Suzuki, *Weak solutions to the Navier-Stokes-Poisson equation*, Adv. Math. Sci. Appl. **18** (2008) 141–168.
173. A. Kubo and T. Suzuki, *Asymptotic behavior of the solution to a parabolic ODE system modeling tumor growth*, Differential and Integral Equations **17** (2004) 721–736.
174. M. Kurokiba and T. Ogawa, *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential and Integral Equations **16** (2003) 427–452.
175. M. Kurokiba, T. Senba, and T. Suzuki, *Type (II) blowup in a smplified system of chemotaxis*, in preparation.
176. M. Kurokiba and T. Suzuki, *On a perturbed system of chemotaxis*, International J. Mathematical Analysis **1** (2007) 1261–1272.
177. T.G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc. **186** (1973) 259–272.
178. J.S. Langer, *Models of pattern formation in first-order phase transitions*, In; *Direction in Condensed Matter Physics*, pp. 164–186, World Scientific Publishers, 1986.

179. H.A. Levine and B.D. Sleeman, *A system of reaction and diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math. **57** (1997) 683–730.
180. Y.Y. Li, *Harnack type inequality: the method of moving planes*, Comm. Math. Phys. **200** (1999) 421–444.
181. Y.Y. Li and I. Shafrir, *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. **43** (1994) 1255–1270.
182. C.-S. Lin, *Uniqueness of solutions to the mean field equations for the spherical Onsager vortex*, Arch. Rational Mech. Anal. **153** (2000) 153–176.
183. C.-S. Lin and M. Lucia, *Uniqueness of solutions for a mean field equation on torus*, J. Differential Equations **229** (2006) 172–185.
184. J.-L. Lions, *Quelques Méthodes de Résolution de Problèmes aux Limites Non Linéaires*, Dunod-Gauthier-Villas, Paris, 1969.
185. P.-L. Lions, *Structure of the set of steady-state solutions and asymptotic behavior of semilinear heat equations*, J. Differential Equations **53** (1984) 362–386.
186. P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol.1, Incompressible Models*, Clarendon Press, Oxford, 1996.
187. P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. 2, Compressible Models*, Clarendon Press, Oxford, 1998.
188. P.-L. Lions and G. Toscani, *Diffusive limit for finite velocity Boltzmann kinetic models*, Rev. Mat. Iberoamericana **13** (1997) 473–513.
189. K.E. Lonngren and A. Hirose, *Expansion of an electron cloud*, Phys. Lett. A **59** (1976) 285–286.
190. S. Luckhaus and S. Zheng, *A nonlinear boundary value problem involving a nonlocal term*, Nonlinear Analysis **176** (1994) 129–135.
191. L. Ma and J.C. Wei, *Convergence of Liouville equation*, Comment. Math. Helv. **76** (2001) 506–514.
192. A.J. Majda, *Remarks on weak solutions for vortex sheets with a distinguished sign*, Indiana Univ. Math. J. **42** (1993) 921–939.
193. T. Makino, *On a local existence theorem for evolution equation of gaseous stars*, In; Patterns and Waves (ed. T. Nishida, M. Mimura, and H. Fujii), Stud. Math. Appl. **18**, North-Holland, 1986, pp. 459–479.
194. T. Makino, *Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars*, Transport Theory and Statistical Physics **21** (1992) 615–642.
195. T. Makino and B. Perthame, *Sur les solutions à symétrie sphérique de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuses*, Japan J. Appl. Math. **7** (1990) 165–170.
196. T. Makino and S. Ukai, *Sur l'existence de solutions locales de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuses*, J. Math. Kyoto Univ. **27** (1987) 387–399.
197. T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'équation d'Euler compressible*, Japan J. Appl. Math. **3** (1986) 249–257.
198. H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18** (1978) 221–227.
199. H. Matano, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. RIMS. Kyoto Univ. **15** (1979) 401–454.
200. H. Matano, *Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation*, J. Fac. Sci. Univ. Tokyo, Sec. IA **29** (1982) 401–441.
201. H. Matano, *Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems*, J. Fac. Sci. Univ. Tokyo, Sec. IA **30** (1984) 645–673.
202. I. Miyadera, *Nonlinear Semigroups*, Amer. Math. Soc., Providence, 1992.
203. I. Miyasita, *On the dynamical properties for Fix-Caginalp equation*, preprint.
204. N. Mizoguchi and T. Suzuki, *Equations of gas combustion: S-shaped bifurcation and mush-*

rooms, J. Differential Equations **134** (1997) 183–215.

205. L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987) 123–142.
206. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971) 1077–1092.
207. L. Mou and C. Wang, *Bubbling phenomena of Palais-Smale-like sequences of m-harmonic type systems*, Calc. Var. **4** (1996) 341–367.
208. M. Murakami, K. Nishihara, and T. Hanawa, *Self-gravitational collapse of radially cooling spheres*, Astrophys. J. **607** (2004) 879–889.
209. T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. **5** (1995) 581–601.
210. T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains*, J. Inequality and Applications **6** (2001) 37–55.
211. T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997) 411–433.
212. T. Nagai, T. Senba, and T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, Hiroshima Math. J. **30** (2000) 463–497.
213. K. Nagasaki and T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*, Asymptotic Analysis **3** (1990) 173–188.
214. J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. I.R.E. **50** (1962) 2061–2070.
215. Y. Naito and T. Suzuki, *Self-similarity in chemotaxis systems*, Colloquium Mathematics **111** (2008) 11–34.
216. Y. Naito, T. Suzuki, and K. Yoshida, *Self-similar solutions to a parabolic system modeling chemotaxis*, J. Differential Equations **184** (2002) 386–421.
217. V. Nanjundiah, *Chemotaxis, signal relaying, and aggregation morphology*, J. Theor. Biol. **42** (1973) 63–105.
218. Z. Nehari, *On the principal frequency of a membrane*, Pacific J. Math. (1958) 285–293.
219. Z. Nehari, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. **95** (1960) 101–123.
220. Z. Nehari, *Characteristic values associated with a class of nonlinear second-order diffential equations*, Acta Math. **105** (1961) 141–175.
221. C. Neri, *Statistical mechanics and the n-point vortex system with random intensities on a bounded domain*, Ann. Inst. Henri Poincaré, Analyse non linéaire **21** (2004) 381–399.
222. W.-M. Ni, *Qualitative properties of solutions to elliptic problems*, In: Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 1 (ed. M. Chipot and P. Quittner), Elsevier, Amsterdam, 2005, pp. 157–233.
223. W.-M. Ni, P.E. Sacks, and J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations **54** (1984) 97–120.
224. G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems - From Dissipative Structures to Order through Fluctuations*, John Wiley and Sons, New York, 1977.
225. H. Nielsen and P. Olesen, *Vortex-line models for dual strings*, Nucl. Phys. B **61** (1973) 45–61.
226. Y. Nishiura, *Global structure of bifurcating solutions of some reaction-diffusion systems*, SIAM J. Math. Anal. **13** (1982) 555–593.
227. Y. Nishiura, *Far-from-Equilibrium Dynamics*, Amer. Math. Soc., Providence, 2002.
228. Y. Nishiura and I. Ohnishi, *Some mathematical aspects of the mocro-phase separation in diblock coplymers*, Physica D **84** (1995) 31–39.
229. M. Obata, *The conjecture on conformal transformations of Riemannian manifolds*, J. Differential Geometry **6** (1971) 247–253.
230. T. Ohta, D. Jansnow, and K. Kawasaki, *Universal scaling in the motion of random interfaces*,

Phys. Rev. Lett. **49** (1982) 1223–1226.
231. T. Ohta and K. Kawasaki, *Equilibrium morphology of block copolymer melts*, Macromolecules **19** (1986) 2621–2632.
232. H. Ohtsuka, T. Ricchiardi, and T. Suzuki, *A Trudinger-Moser inequality associated with a point vortex meand field equation*, preprint.
233. H. Ohtsuka, T. Senba, and T. Suzuki, *Blowup in infinite time in the simplified system of chemotaxis*, Adv. Math. Sci. Appl. **17** (2007) 445–472.
234. H. Ohtsuka and T. Suzuki, *Palais-Smale sequence relative to the Trudinger-Moser inequality*, Calc. Vari. in PDE **17** (2003) 235–255.
235. H. Ohtsuka and T. Suzuki, *Blow-up analysis for Liouville type equation in self-dual gauge field theories*, Comm. Contemp. Math. **7** (2005) 177–205.
236. H. Ohtsuka and T. Suzuki, *Mean field equation for the equilibrium turbulence and related functional inequality*, Adv. Differential Equations **11** (2006) 281–304.
237. O.A. Oleinik, *A method of solution of the general Stefan problem* (in Russian), Dokl. Acad. Nauk. SSSR **135** (1960) 1054–1057, translated as; Soviet Math. Dokl. **1** (1960) 1350–1354.
238. V. Oliker, *Embedding S^n into $\mathbf{R}^{n+1}$ with given integral Gauss curvature and optimal mass transport theory on S^n*, Adv. Math. **213** (2007) 600–620.
239. K. Ono and W. Strauss, *Regular solutions of the Vlasov-Poisson-Fokker-Planck system*, Discrete and Continuous Dynamical Systems **6** (2000) 751–772.
240. E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982) 321–326.
241. L. Onsager, *Statistical hydrodynamics*, Suppl. Nuovo Cimento **6** (1949) 279–287.
242. S. Osher and A. Sethian, *Fronts propagating with curvature dependent speed: algorithm based on Hamilton-Jacobi formulations*, J. Comp. Phys. **79** (1988) 12–49.
243. Y. Oshita, *On stable nonconstant stationary solutions and mesoscopic patterns for FitzHugh-Nagumo equations in higher dimensions*, J. Differential Equations **188** (2003) 110–134.
244. Y. Oshita, *Stable staionary patterns and interfaces arising in reaction-diffusion systems*, SIAM J. Math. Anal. **36** (2004) 479–497.
245. M. Otani, *Existence and asymptotic stability of strong solutions of nonlinear evolution equations with a difference term of subdifferentials*, Colloq. Math. Soc. Janos Bolyai **30**, Qualitative Theory of Differential Equations, North-Holland, Amsterdam, 1980.
246. H.G. Othmer and T. Hillen, *The diffusion limit of transport equations II: Chemotaxis equations*, SIAM J. Appl. Math. **62** (2002) 1222–1250.
247. H.G. Othmer and A. Stevens, *Aggregation, blowup, and collapse: the ABC's of taxis in reinforced random walk*, SIAM J. Appl. Math. **57** (1997) 1044–1081.
248. C.S. Patlak, *Random walk with persistence and external baias*, Bull. Math. Biophys. **15** (1953) 311–338.
249. I. Pawlow, *Three-dimensional model of thermomechanical evolution of shape memory materials*, Control and Cybernetics **29** (2000) 341–365.
250. I. Pawlow and A. Zochowski, *Existence and uniqueness of solutions for a three-dimensional thermoelastic system*, Dissertationes Mathematicae **406** (2002) 1–46.
251. R.L. Pego, *Front migration in the nonlinear Cahn-Hilliard equation*, Proc. Roy. Soc. London **A422** (1989) 261–278.
252. O. Penrose and P.C. Fife, *Thermodynamically consistent models of phase-field type for the kinetics of phase transition*, Physica D **43** (1990) 44–62.
253. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math. DG/021159.
254. G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math. DG/0303109.
255. G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math. DG/0307245.

256. S.I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965) 1408–1411.
257. S.I. Pohozaev, *On the Sobolev imbedding theorem for* $p\ell = n$, Doklady Conference, Section Math. Moscow Power Inst. (1965) 158–170.
258. Y.B. Pointin and T.S. Lundgren, *Statistical mechanics of two-dimensional vortices in a bounded container*, Phys. Fluids **19** (1976) 1459–1470.
259. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.
260. J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. Anal. Geom. **3** (1995) 297–315.
261. J. Qing, *Harmonic maps from surfaces*, In; *Nonlinear Functional Analysis and Applications to Differential Equations*, (ed. A. Ambrosetti, K. C. Chang, and I. Ekeland), World Sci. Publ., River Edge, 1998, pp. 203–236.
262. J. Qing and G. Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math. **50** (1997) 295–310.
263. P. Quittner and P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhäuser, Basel, 2007.
264. M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.
265. M. Rascle, *Sur une équation intégro-différentielle non linéaire issue de la biologie*, J. Differential Equations **32** (1979) 420–453.
266. J. Rubinstein, P. Sternberg, and J.B. Keller, *Fast reaction, slow diffusion and curve shortening*, SIAM J. Appl. Math. **49** (1989) 116–133.
267. S. Sakaguchi and T. Suzuki, *Interior imperfect ignition can not occur on a set of positive measure*, Arch. Rational Mech. Anal. **142** (1998) 143–153.
268. A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equation*, Walter de Gruyter, Berlin, 1995.
269. K. Sawada and T. Suzuki, *Derivation of the equilibrium mean field equations of point vortex system and Vortex filament system*, Theoretical and Applied Mechanics Japan, **56** (2008) 285–290.
270. R. Schaaf, *Global behaviour of solution branches for some Neumann problems depending on one or several parameters*, J. Reine Ang. Math. (1984) 1–31.
271. R. Schaaf, *Stationary solutions of chemotaxis systems*, Trans. Amer. Math. Soc. **292** (1985) 531–556.
272. S. Schochet, *The weak vorticity formulation of the* $2-D$ *Euler equations and concentration-cancellation*, Comm. Partial Differential Equations **20** (1995) 1077–1104.
273. L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
274. T. Senba, *Blow-up of radially symmetric solutions to some systems of partial differential equations modelling chemotaxis*, Adv. Math. Sci. Appl. **7** (1997) 79–92.
275. T. Senba, *Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domains*, Funkcial. Ekvac. **48** (2005) 247–271.
276. T. Senba, *Type II blowup of solutions to a simplified Keller-Segel system in two dimensions*, Nonlinear Analysis **66** (2007) 1817–1839.
277. T. Senba and T. Suzuki, *Some structures of the solution set for a stationary system of chemotaxis*, Adv. Math. Sci. Appl. **10** (2000) 191–224.
278. T. Senba and T. Suzuki, *Parabolic system of chemotaxis; blowup in a finite and in the infinite time*, Meth. Appl. Anal. **8** (2001) 349–368.
279. T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations **6** (2001) 21–50.
280. T. Senba and T. Suzuki, *Time global solutions to a parabolic-elliptic system modelling chemotaxis*, Asymptotic Analysis **32** (2002) 63–89.

281. T. Senba and T. Suzuki, *Blowup behavior of solutions to re-scaled Läger-Luckhaus system*, Adv. Differential Equations **8** (2003) 787–820.
282. T. Senba and T. Suzuki, *Applied Analysis: Mathematical Methods in Natural Science*, Imperial College Press, London, 2004.
283. I. Shafrir, *A* $\sup + \inf$ *inequality for the equation* $-\Delta u = Ve^u$, C. R. Acad. Sci. Paris, **315**, Série I. (1992) 159–164.
284. I. Shafrir and G. Wolansky, *The logarithmic HLS inequality for systems on compact manifolds*, J. Funct. Anal. **227** (2005) 200–226.
285. I. Shafrir and G. Wolansky, *Moser-Trudinger and logarithmic HLS inequalities for systems*, J. Euro. Math. Soc. **7** (2005) 413–448.
286. T.C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985) 475–485.
287. H.L. Smith, *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, Ameth. Math. Soc., Providence, 1995.
288. E.M. Stein, *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.
289. P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch Rational Mech. Anal. **101** (1988) 209–260.
290. A. Stevens, *The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-partile systems*, SIAM J. Appl. Math. **61** (2000) 183–212.
291. B.E.E. Stoh, *Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry*, J. Differential Equations **125** (1996) 154–183.
292. N. Straumann, *General Relativity and Relativistic Astrophysics*, Springer-Verlag, Berlin, 1991.
293. D.W. Stroock, *Some stochastic processes which arise from a model of the motion of a bacterium*, Probability Theory and Related Fields **28** (1974) 301–315.
294. M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984) 511–517.
295. M. Struwe, *On the evolution of harmonic mapping of Riemannian surfaces*, Comment. Math. Helv. **60** (1985) 558–581.
296. M. Struwe, *Variational Methods, Application to Partial Differential Equations and Hamiltonian Systems*, third ed., Springer-Verlag, Berlin, 2000.
297. M. Struwe and G. Tarantello, *On multivortex solutions in Chern-Simons gauge theory*, Boll. U.M.I. Sez. B (8) **1** (1998) 109–121.
298. Y. Sugiyama, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Differential and Integral Equations **19** (2006) 841–876.
299. Y. Sugiyama, *Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis*, Differential and Integral Equations **20** (2007) 133–180.
300. Y. Sugiyama, *Partial regularity and blow-up asymptotics of weak solutions to degenerate parabolic system of porous medium type*, preprint.
301. T. Suzuki, *Global analysis for a two-dimensional elliptic eigenvalue problem with exponential nonlinearity*, Ann. Inst. Henri Poincaré, Analyse non linéaire **9** (1992) 367–398.
302. T. Suzuki, *A note on the stability of stationary solutions to a system of chemotaxis*, Comm. Contemporary Math. **2** (2000) 373–383.
303. T. Suzuki, *Semilinear Elliptic Equations*, Gakkotosho, Tokyo, 1994.
304. T. Suzuki, *Free Energy and Self-Interacting Particles*, Birkhüser, Boston, 2005.
305. T. Suzuki, *Method of the blowup envelope and applications*, Advanced Studied in Pure Mathematics **47-2**, Asymptotic Analysis and Singularities, Elliptic and parabolic PDEs and related problems, pp. 519–542, Math. Soc. Japan, Tokyo, 2007.
306. T. Suzuki, *Variational and scaling properties of self-interacting fluids and their mass quantiza-*

tions, Theoretical and Applied Mechanics Japan, **56** (2008) 207–213.
307. T. Suzuki, *Semilinear parabolic equation on bounded domain with critical Sobolev exponent*, to appear in; Indiana Univ. Math. J..
308. T. Suzuki and F. Takahashi, *Nonlinear eigenvalue problem with quantization*, Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 5 (ed. M. Chipot), pp. 277–370, Elsevier, Amsterdam, 2008.
309. T. Suzuki and F. Takahashi, *Capacity estimate for the blow-up set of parabolic equations*, to appear in; Math. Z..
310. T. Suzuki and R. Takahashi, *Global in time solution to a class of tumour growth systems*, preprint.
311. T. Suzuki and R. Takahashi, *Degenerate parabolic equation with critical exponent derived from the kinetic theory*, preprint.
312. T. Suzuki and S. Tasaki, *Stationary Fix-Caginalp equation with non-local term*, preprint.
313. T. Suzuki and S. Yoshikawa, *Stability of the steady state for the Falk model system of shape memory alloys*, Math. Meth. Appl. Sci. **30** (2007) 2233–2245.
314. H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.
315. G. Tarantello, *Mutiple condensate solutions for the Chern-Simons-Higgs theory*, J. Math. Phys. **37** (1996) 3769–3796.
316. G. Tarantello, *Analytic aspects of Liouville-Type equations with singular sources*, Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 1 (ed. M. Chipot and P. Quittner), Elsevier, Amsterdam, 2004, pp. 491–592.
317. C.H. Taubes, *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations*, Comm. Math. Phys. **72** (1980) 277–292.
318. C.H. Taubes, *On the equivalence of the first and second order equations for gauge theories*, Comm. Math. Phys. **75** (1980) 207–227.
319. R. Temam, *A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma*, Arch. Rational Mech. Anal. **60** (1975) 51–73.
320. R. Temam, *Remarks on a free boundary value problem arsing in plasma physics*, Comm. Partial Differential Equations **2** (1977) 563–585.
321. R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam, 1984.
322. J.F. Toland, *Duality in nonconvex optimization*, J. Math. Anal. Appl. **66** (1978) 399–415.
323. J.F. Toland, *A duality principle for non-convex optimization and the calculus of variations*, Arch. Rational Mech. Anal. **71** (1979) 41–61.
324. P. Topping, *Winding behaviour of finite-time sigularities of the harmonic map heat flow*, Math. Z. 247 (2004) 279–302.
325. N.S. Trudinger, *On imbedding into Orlicz space and some applications*, J. Math. Mech. **17** (1967) 473–484.
326. J.L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations*, Oxford University Press, Oxford, 2006.
327. J.L. Vázquez, *Evolution of point masses by planar logarithmic diffusion: Finite-time blow-down*, Discrete and Continuous Dynamical Systems **19** (2007) 1–35.
328. H. D. Victory, Jr., *On the existence of global weak solutions to Vlasov-Poisson-Fokker-Planck systems*, J. Math. Anal. Appl. **160** (1991) 525–555.
329. A. Visintin, *Models of Phase Transitions*, Birkhäuser, Boston, 1996.
330. C. Y. Wang, *Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets*, Houston J. Math. **22** (1996) 559–590.
331. G. Wang and D. Ye, *On a nonlinear elliptic equation arising in a free boundary problem*, Math. Z. **244** (2003) 531–548.
332. S. Wang, *Some nonlinear elliptic equations with subcritical growth and critical behavior*, Houston J. Math. **16** (1990) 559–572.

333. J. Wei, *Multiple condensations for a semilinear elliptic equation with sub-critical growth and critical behavior*, Proc. Edinburgh Math. Soc. **44** (2001) 631–660.
334. J. Wei and M. Wintner, *Stability of monotone solutions for the shadow Gierer-Meihardt system with finite diffusivity*, Differential and Integral Equations **16** (2003) 1153–1180.
335. J. Wei and M. Wintner, *Existence and stability analysis of symmetric patterns for the Gierer-Meinhardt system*, J. Math. Pures Appl. **83** (2004) 433–476.
336. M.B. Williams and S.H. Davis, *Nonlinear theory of film rupture*, J. Colloidal and Interface Sci. **90** (1982) 220–228.
337. E. Witten, *Some exact multipseudoparticle solutions of classical Yang-Mills theory*, Phys. Rev. Lett. **38** (1977) 121–124.
338. G. Wolansky, *On the evolution of self-attracting clusters and applications to semilinear equations with exponential nonlinearity*, J. Anal. Math. **59** (1992) 251–272.
339. G. Wolansky, *On steady distributions of self-attracting clusters under friction and fluctuations*, Arch. Rational Meth. Anal. **119** (1992) 355–391.
340. G. Wolansky, *Critical behavior of semi-linear elliptic equations with sub-critical exponents*, Nonlinear Analysis **26** (1996) 971–995.
341. G. Wolansky, *A critical parabolic estimate and application to nonlocal equations arising in chemotaxis*, Applicable Analysis **66** (1997) 291–321.
342. G. Wolansky, *A concentration theorem for the heat equation*, Monatsh. Math. **132** (2001) 255–261.
343. M. Wolfrum, *Geometry of heteroclinic cascades in scalar parabolic differential equations*, J. Dynamics and Differential Equations **14** (2002) 207–241.
344. A. Yagi, *Norm behavior of solutions to the parabolic system of chemotaxis*, Math. Japonica, **45** (1997) 241–265.
345. T. Yamaguchi, N. Suematsu, and H. Mahara, *Self-organizing hierarchy: dissipaive structure assisted self-assembly of metal nanoparticles in polymer matrices*, ACS Symp. Ser. **869** (2004) 16–27.
346. Y. Yanagida, *Reaction-diffusion systems with skew-gradient structure*, Meth. Appl. Anal. **8** (2001) 209–226.
347. Y. Yanagida, *Mini-maximizers for reaction-diffusion systems with skew-gradient structure*, J. Differential Equations **179** (2002) 311–335.
348. Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer-Verlag, New York, 2001.
349. Y. Yang, H. Chen, and W. Liu, *On existence and non-existence of global solutions to a system of reaction-diffusion equations modeling chemotaxis*, SIAM J. Math. Anal. **33** (1997) 763–785.
350. R. Ye, *Global existence and convergence of the Yamade flow*, J. Differential Geometry **39** (1994) 35–50.
351. S. Yoshikawa, *Weak solutions for the Falk model system of shape memory alloys in energy class*, Math. Meth. Appl. Sci. **28** (2005) 1423–1443.
352. S. Yoshikawa, *Unique global existence for a three-dimensional thermoelastic system of shape memory alloys*, Adv. Math. Sci. Appl. **15** (2005) 603–627.
353. S. Yoshikawa, P. Pawlow, and W.M. Zajaczkowski, *Quasilinear thermoelasticity system arising in shape memory materials*, SIAM J. Math. Anal. **38** (2007) 1733–1759.
354. T.J. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4** (1968) 17–22.
355. S. Zheng, *Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems*, Longman, New York, 1995.
356. H. Zou, *Symmetry of positive solutions to* $\Delta u + u^p = 0$ *in* $\mathbf{R}^n$, J. Differential Equations **120** (1995) 46–88.

Index